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GRAPHICAL ANALYSES OF NONLINEAR CIRCUITS*

BY

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1. Introduction. As a general rule, a problem in physics is not considered solved unless the solution can be expressed in analytical form. The same usually holds true in the case of engineering problems, although there the art often progresses faster than the theory under the impact of economic forces, and the engineer is often forced to seek a solution by means of an experimental setup, or possibly by means of some numerical or graphical process.

The disadvantage of a numerical or graphical method is its lack of generality, its tendency towards inaccuracy, particularly owing to cumulative errors, and its inability to exhibit optimum values for the parameters involved, particularly if these have to be in numerical rather than in symbolic form. On the other hand, these methods often yield answers to problems that the analytical method cannot handle, and furthermore are often very effective as teaching aids. This is particularly true of the graphical methods.

It is the purpose of this article to illustrate the application of graphical constructions to problems involving nonlinear circuits, particularly those containing vacuum tubes. It is the writer's hope that some mathematician will be sufficiently attracted to this method to attempt to establish it on a more general basis, possibly something akin to the collection of theorems of ordinary Euclidean or of Projective Geometry.

2. Definition of graphical method. Before proceeding with a description of the method it will be desirable to define it. By graphical constructions are meant those geometrical *manipulations* by which a solution to a problem is obtained. It may be necessary to slide a curve representing a relationship between two variables along the axis of the independent variable, and to find (geometrically) where it intersects another curve representing a second relationship between the two variables. The manipulations may be more involved than those of simple translation along the axis, and it is to be stressed that the restriction of ruler and compass constructions is not invoked in these manipulations.

It is apparent that the method is not that usually understood by the average engineer, namely, the plotting of a complicated generalized analytical expression to permit values to be taken off the graph in order to obviate the need for computing the value of the expression every time the problem arises.

3. Simple series nonlinear circuit. As an elementary example of a graphical construction, let us consider the circuit shown in Fig. 1, that of a diode (two-element

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vacuum tube) in series with a resistance R and a source of d.c. potential E . It is desired to find the current flow in this circuit.

It is necessary to know the voltage-current $e-i$ relationship for the diode, and for the resistor R . We assume for simplicity that the latter is a linear resistance. Then the $e-i$ relationship is that shown in Fig. 2. The curve is a straight line making an angle θ with the voltage axis, such that

$$\cot \theta = R, \quad (1)$$

the resistance of the device. This slope is constant and hence R has a fixed value: so many volts per ampere, or ohms.

On the other hand, the diode has the characteristic shown in Fig. 3. Here, for

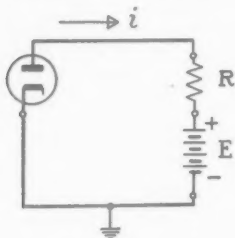


FIG. 1.

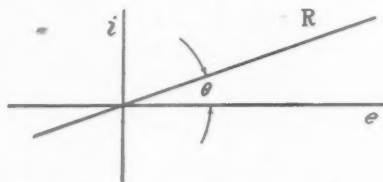


FIG. 2.

negative values of voltage (plate negative to cathode) no current can flow; while for positive values of voltage current flows in such manner as to generate the curve shown. The ideal diode would have the following equation for *positive* plate voltages

$$i = Ae^{3/2}e, \quad (2)$$

but actual diodes depart to some extent from the above equation owing to such factors as initial velocity of emission of electrons from the cathode, the effect of the supporting members for the cathode and plate, etc.

The diode is a nonlinear device; first because of the break in the curve at the origin and second because even for positive plate voltages the $e-i$ relationship is usually not a straight line. One can define the resistance as

1) the reciprocal slope of the secant line to any point of the curve (this is the so-called d.c. resistance) or

2) the reciprocal slope of the tangent line to any point of the curve (this is usually called the a.c., incremental, or variational resistance of the device).

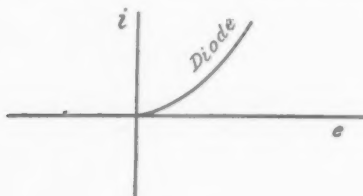


FIG. 3.

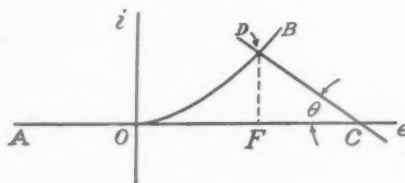


FIG. 4.

Such concepts have limited utility however, since the resistance in either case is no longer a constant, but a function of the applied voltage or current through the device.

The graphical method to be described takes the fundamental $e-i$ relationship, or terminal characteristic as it has been called by Kirschstein, and operates directly with it. Furthermore, the curve does not have to be analytic, nor even expressible in the form of an equation; it can simply be a plot of experimental data, although this involves interpolation between experimentally determined points.

The process of finding the current through the two in series for the impressed voltage E is essentially that of solving the two equations for the terminal characteristics simultaneously under the condition that the sum of the voltage drops across the two elements must equal the impressed voltage E . Thus, if the relationship for the one element is $i=f_1(e)$, then that for the other is $i=f_2(E-e)$, and it is desired to find a common value of i that satisfies both relationships.

Since one or both of the above equations may be of degree higher than unity, the analytical solution cannot be effected by the method of determinants, but rather by the method of substitution, and finally results in the necessity for solving an equation of degree higher than unity.

This, however, assumes that terminal characteristics can be represented by power series. The graphical method requires no such condition; it operates on the graphical plots directly. Thus, suppose the terminal characteristic of the diode is represented by AOB, Fig. 4. Let OC represent the magnitude of the impressed voltage E . Through C draw DC at an angle θ , as shown, such that $\cot \theta = R$. Then the intersection of CD and AOB in D represents the required solution, in that DF is the common current in this series circuit; OF is the voltage drop in the diode; FC is the voltage drop in the resistor R ; and clearly $OF+FC$ equals the impressed voltage E . If E varies with time, DC can be shifted back and forth along the voltage axis at positions corresponding to the instantaneous values of E , and the intersections will furnish the corresponding instantaneous values of the current.

The above solution represents a well-known method for solving two equations simultaneously when the equations are of degree higher than the first or even of transcendental nature. It will be of interest, however, to see how this method is applied to a more complicated circuit.

4. Triode tube and resistance in series. The next example will be that of a three-element or triode tube in series with a resistance and a source of d.c. voltage E_{bb} . The electrical connections are shown in Fig. 5. The additional complication is that in the triode the plate current is a function of two variables; the grid voltage and the plate voltage. The terminal characteristic must therefore be represented by a three-dimensional plot involving the plate current i_p , the grid voltage e_g (which is the sum of the instantaneous value of the alternating signal voltage e_s and the constant, d.c. bias voltage E_c), and the plate voltage e_p .

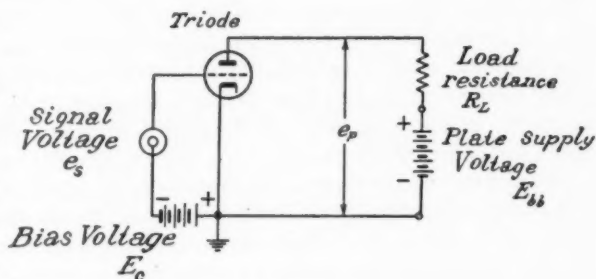


FIG. 5.

The resulting plot is a curved surface in space. It can be represented in two dimensions by a family of curves which represent discrete projections of this surface upon any one of the three coordinate planes. For the problem at hand the most useful set of projections is that upon the e_p-i_p coordinate plane, in the form of a family of e_p-i_p curves with e_g as the parameter. This is shown in Fig. 6 (solid lines). Curves for which e_g is positive have been omitted for simplicity.

Assume further for simplicity that R_L , the load resistance, is linear. The current through it is a function of but one voltage, that which must be applied across its terminals to produce the above current flow. To represent its terminal characteristic

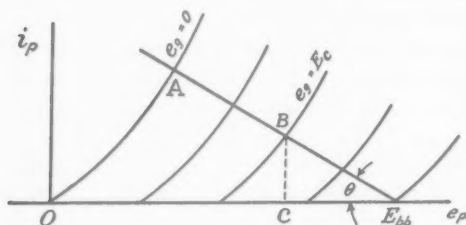


FIG. 6.

The straight line makes an angle θ with the e_p axis such that $\cot \theta = R_L$, i.e., the R_L plane is inclined at the angle θ to the e_g-e_p coordinate plane.

The graphical solution consists in drawing the line of intersection EA at the angle θ to the e_p axis. The intersection of EA with the tube family of curves gives the common value of current flowing through the plate circuit of the triode and R_L in series, for any given value of grid voltage e_g . For example, at a moment when the signal voltage e_s is passing through zero, the instantaneous value of the grid voltage e_g is simply that of the bias battery, E_c . The instantaneous value of the plate current is BC, where B is the intersection of AC with that curve of the plate family for which $e_g = E_c$. It is further to be noted that the instantaneous plate voltage e_p is OC, and the instantaneous value of the voltage drop across R_L is EC.

For other instantaneous values of e_g , other curves of the plate family are involved, and the process of determining the instantaneous values of plate current, plate voltage, and load voltage (across R_L) is identical to that described above. Thus, for a signal voltage impressed upon the input or grid circuit, the output signal voltage between the plate and ground can be found. Such matters as the amplification of the stage, distortion in the output, etc., can then be determined.

In passing, we may note here that the locus of the plate current for various values of e_g is the intersection of the tube surface and the R_L plane in space. This intersection is a curve in space, but fortunately its projection on the e_p-i_p plane is a straight line, namely the intersection of the R_L plane itself with the e_p-i_p plane. It is for that reason that the e_p-i_p family of the tube curves is employed; the graphical construction is simply the points of intersection of a straight line representing R_L with the above plate family.

The above problem can become much more complicated under certain conditions. For example, if the input signal voltage is great enough, the grid can be driven posi-

in three dimensions, it is plotted as a plane whose intersection with the e_p-e_g coordinate plane is a straight line parallel to the e_g axis. In this way the current in it is independent of the e_g coordinate, and is a (linear) function of but one voltage, that corresponding to the plate voltage e_p of the tube. All points of this plane representing R_L project over to the i_p-e_p coordinate plane as a straight line that is also the intersection of the above R_L plane with the i_p-e_p plane.

tive with respect to the cathode, whereupon it draws current during the positive peak of the a.c. cycle. If the signal source has appreciable internal impedance, then a voltage drop will occur in the source during the above portion of the cycle, and the actual voltage applied to the grid will differ from the generated voltage e_s .

It is therefore necessary to determine the actual grid voltage before the plate current can be found. Another complication arises however, in that the grid current (and hence the actual grid voltage) is a function not only of the positive grid voltage, but of the plate voltage as well. This is because the space current divides between the two electrodes in a manner depending upon the two electrode voltages. At the same time the plate voltage is a function of R_L and the grid voltage. Thus the above simple graphical construction can become quite involved if merely the input signal is increased to a point where the grid is driven positive.

5. The balanced amplifier. Instead of investigating such details, important though they may be, it will be of interest to examine another type of circuit very important in the communication industry. Reference is made to the push-pull or balanced amplifier. The circuit is shown in Fig. 7.

In (A) is shown the actual circuit, whereas in (B) is shown an idealization or equivalent form better suited for the purpose of analysis. In the actual circuit (A), two tubes are employed, inductively coupled to each other and the output load resistance r_L by an output transformer. The signal on one grid is 180 degrees out of phase with that on the other grid, as is suggested by the symbols $+e_s$ and $-e_s$. The bias voltage E_c , on the other hand, is applied to both grids in the same polarity; and the plate supply voltage is applied to the two tubes in the same polarity too, as shown.

The actual load resistance r_L and the output transformer can be replaced by the center-tapped inductance and reflected load resistance R_L as far as the tubes are concerned. The simplified circuit is shown in (B), Fig. 7. In using this equivalent circuit, it is tacitly assumed that the actual output transformer is an ideal transformer having infinite primary and secondary open-circuit inductance, no distributed capacity, unity coefficient of coupling between windings, etc. In the equivalent circuit the center-tapped inductance is assumed to be infinite in value and to have unity coupling between the two halves of the complete winding. Ordinarily this is a reasonable assumption.

As a result, the current in one-half of the winding cannot at any moment exceed that in the other half for otherwise an infinite counter-electromotive force would be induced in the windings that would tend to prevent such an inequality from taking

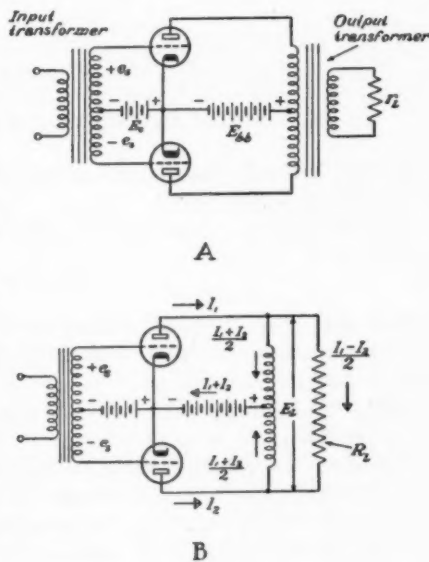


FIG. 7.

place. The currents in the windings can vary, however, provided they remain equal to one another at all times. Finally, two further assumptions are made, namely, that the signal voltages e_s and $-e_s$ applied to the two grids are at all times equal and opposite to one another, and that the two tubes have identical terminal characteristics. These two assumptions seem also reasonable.

Consider first that e_s equals zero (no signal is applied). The bias on each grid is E_c , and the plate voltage for either tube is E_{bb} , hence the two plate currents I_1 and I_2 are equal to one another. Since they flow in opposite directions from the ends of the winding to the center tap, they balance each other magnetically in the output inductance and produce no voltage across the ends. Consequently no current flows in the load resistance R_L .

Now suppose that a signal voltage is impressed such that the top grid is driven positive by an amount e_{s1} from its normal d.c. negative bias value of E_c , and that the bottom grid is driven more negative by an equal amount, i.e., $-e_{s1}$. The two plate currents will now vary in opposite directions, namely, I_1 will increase and I_2 will decrease. However, the sum of these two currents flows through the plate power supply, and owing to the infinite inductance of the center-tapped winding, $(I_1 + I_2)/2$ flows down through the top half of the winding, and an equal amount flows up through the bottom half, to combine at the center tap to furnish the sum $(I_1 + I_2)$ flowing through the power supply.

Since $(I_1 + I_2)/2$ is the average between I_1 and I_2 , it is equal to neither, and from the principle of continuity of current flow, the difference

$$I_1 - \frac{1}{2}(I_1 + I_2) = \frac{1}{2}(I_1 + I_2) - I_2 = \frac{1}{2}(I_1 - I_2) \quad (3)$$

must flow through R_L . A quick check will indicate that Kirchhoff's current law is satisfied at each junction.

The current $(I_1 - I_2)/2$ is the output current. In flowing through R_L , it sets up a voltage drop

$$E_L = \frac{1}{2}(I_1 - I_2)R_L. \quad (4)$$

Half of this or $E_L/2$ appears across each half of the output winding of such polarity that the instantaneous plate voltage of the top tube is $E_{bb} - (E_L/2)$ and that of the bottom tube is $E_{bb} + (E_L/2)$.

Thus the following facts have been brought to light:

- 1) The grid voltages change by equal but opposite increments from their common bias value E_c owing to the center tap on the input transformer secondary.
- 2) The plate voltages change by equal but opposite increments from their common supply value E_{bb} owing to the center tap on the output inductance. Moreover, the plate voltage increments are opposite in sign to the corresponding grid voltage increments.
- 3) The plate currents change in opposite directions in the same sense as the corresponding grid voltages, but not necessarily to an equal degree. If the tubes are nonlinear, as is usually the case, then the increase in plate current of either tube for a positive increment in grid voltage is not necessarily the same as the decrease in plate current for an equal negative increment in grid voltage.

From the above facts several graphical constructions are available to determine the plate current and plate voltage variations in the tubes, the output current and

voltage, the power output, and the d.c. power input. The following graphical method is preferred by the author. In Fig. 8 is shown the plate family of curves for either tube. If there is no signal input, the only voltages present are the d.c. potentials E_{bb} applied to the two plates and E_c applied to the two grids. The current through either tube is then $I_b = E_{bb}B$, a direct current.

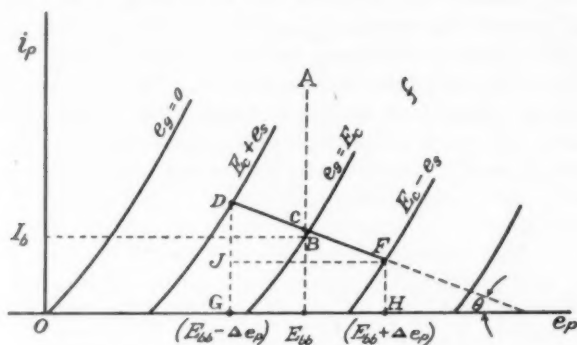


FIG. 8.

Now suppose that equal and opposite signal voltages e_s and $-e_s$ are applied to the grids in addition to E_c . Then the current in the one tube will increase from BE_{bb} to DG , and that in the other tube will drop to FH , as shown. The plate voltage of the first tube will drop from OE_{bb} to $OG = (E_{bb} - \Delta e_p)$, and that in the other tube will rise by an equal amount to $OH = (E_{bb} + \Delta e_p)$.

It is also clear from Fig. 8 that DJ represents the difference between the two currents or $(I_1 - I_2)$, and JF represents $2\Delta e_p$, the voltage across the output inductance and previously denoted by E_L in Fig. 7. From Eq. (4), it is evident that

$$JF/DJ = E_L/(I_1 - I_2) = R_L/2. \quad (5)$$

Thus DF makes the angle θ with the e_p axis such that

$$\cot \theta = R_L/2. \quad (6)$$

It is also evident from the geometry of the figure that $DC = CF$, i.e., that the ordinate through E_{bb} bisects line DF in C .

The above facts suggest the following method of graphical construction. We hold a rule at the angle θ and slide it up or down until the segment between the desired $e_p - i_p$ curves (corresponding to equal and opposite grid voltage excursions from the bias value E_c) is bisected by the ordinate through E_{bb} . The intersections of the rule with the two $e_p - i_p$ curves gives the two instantaneous values of the two tube currents I_1 and I_2 , corresponding to the signal voltages e_s and $-e_s$ and to the plate load resistance R_L , or rather to $R_L/2$.

Then another pair of equal and opposite grid signal voltages are chosen, and the process repeated. This is continued until as many pairs of instantaneous grid signal voltages have been used as is desired. For a symmetrical signal voltage, such as a sine wave, instantaneous values for only one-quarter of a cycle are required.

When the above graphical construction is performed, there is obtained a curve

on the plate family of curves such as that shown in broken lines ABCDE in Fig. 9. This represents the locus of the current for *either* tube over a cycle of grid signal voltage. It also represents the terminal characteristic for R_L as it appears to *either* tube in the presence of the other tube.

The significance of the last statement is as follows: the two tubes may be regarded as two generators connected to a common load R_L . Owing to their nonlinear characteristics, the tubes do not share the load equally throughout the signal cycle; that tube whose apparent internal resistance is lower takes a greater share of the load, i.e., furnishes more than half of the load current $(I_1 - I_2)/2$ flowing through R_L . As a result, R_L appears as a variable or nonlinear resistance to either tube even though it is actually a linear resistance, and its terminal characteristic on either tube's $e_p - i_p$ family of curves is in itself a curved rather than a straight line.

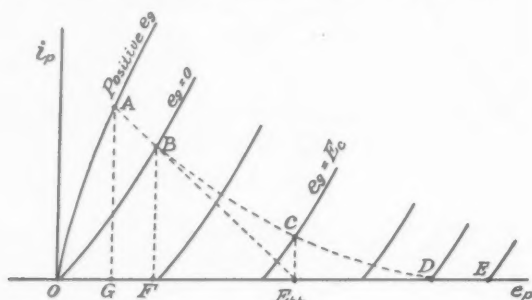


FIG. 9.

Lack of space precludes a detailed discussion of this interesting circuit. However, several important features will be presented. As indicated in Fig. 9, the two $e_p - i_p$ curves passing through B and D, respectively, represent equal and opposite grid swings. The corresponding currents I_1 and I_2 for the two tubes are BF and zero; in short, the tube experiencing the negative grid swing has just reached plate current cutoff.

For $e_p - i_p$ curves passing through A and E, corresponding to a still greater grid swing for either tube, I_1 is AG, and I_2 still remains zero. This means that the second tube is inoperative over this part of the cycle and acts therefore as if it were disconnected. Under these conditions R_L appears to the operative tube as $R_L/4$, which can be expected since the 2 to 1 turns ratio of the output inductance will produce this 4 to 1 impedance transformation if it is unhampered by the other tube.

Portion BA is therefore a straight line whose reciprocal slope corresponds to $R_L/4$. It is easy to show that if it were prolonged, it would pass through E_{bb} . Normally the tubes are operated so that maximum grid signal voltage drives each tube alternately to cutoff or beyond. Maximum output occurs if $R_L/4$ equals either tube's apparent internal plate resistance at the peak of the cycle. The plate resistance of either tube is given by the reciprocal slope of the $e_p - i_p$ curve at point A. Hence a quick determination for the optimum value of R_L , or rather $R_L/4$, is to draw a line through E_{bb} at an angle equal to that of the $e_p - i_p$ curve at point A, and calculate from the reciprocal slope of this line the value of $R_L/4$ and hence of R_L . The complete

characteristic can then be determined by means of the sliding rule as described previously.

Fig. 8 also reveals an interesting point. CE_{bb} is the average between DG and FH, i.e., it represents $(I_1 + I_2)/2$. This is the mid-branch current that flows through the plate supply, as indicated in Fig. 7(B). For various pairs of values of I_1 and I_2 as determined by the sliding rule, the average, or $(I_1 + I_2)/2$ moves up and down along $E_{bb}A$. This is a vertical line or ordinate, and indicates that the resistance to the mid-branch current is zero. This has been tacitly assumed; the output inductance and the plate supply have been assumed to be free of resistance.

If this is not the case, then a line must be drawn through E_{bb} whose reciprocal slope indicates one-half the value of the mid-branch resistance that is present, and the sliding rule must be bisected by this line rather than the ordinate $E_{bb}A$, as is the case in Fig. 8. From this follows several further interesting characteristics.¹

Another point is that not only is the locus of the mid-branch current along the ordinate $E_{bb}A$ in Fig. 8, but that this current executes two alternations per cycle of the grid signal voltage. This means that the mid-branch current is at least double the frequency of the incoming signal; actually, for perfect symmetry, all the even harmonics generated by the tubes flow in parallel through the mid-branch portions of the circuit, while the odd harmonics, including of course the fundamental, flow through the output resistance R_L . Thus, if the tube characteristics are such that the second harmonic is quite prominent, but the third (and higher) harmonics are of small amplitude, then the output wave will be a fairly faithful copy of the input grid signal voltage and the stage will exhibit little distortion. Such a tube characteristic is possessed, for example, by the 6L6 and 807 beam power tubes.

As in the case of the previous constructions for the single-ended tube, various degrees of complication can arise. For example, if the grids are driven positive so that grid current flows, the signal voltage at the grids will be distorted, and this distortion must be determined separately before the above construction can be concluded. Another case is that where the mid-branch plate supply has an internal resistance that is adequately by-passed for the even harmonics, all except the d.c. component. This represents a particularly difficult problem that can be solved only by a series of approximations.

6. Reactive circuits. The previous circuits contained only resistances, linear or nonlinear. If reactances were present, such as the center-tapped output inductance, they were assumed infinite in value and so situated in the circuit as not to have any appreciable a.c. components flowing in them. However, many nonlinear circuits contain reactances of finite value that influence the behavior of the circuit directly, and hence must be taken directly into account.

Owing to lack of space, only the case of an inductance in series with a nonlinear resistance and an a.c. source will be discussed here. Consider the circuit shown in Fig. 10. Here a source of a.c. voltage e is in series with a nonlinear resistance r and inductance L . The voltage e is a known function of time, and the terminal characteristic for r and the value of L is given. It is desired to find the current flow in this circuit.

¹ See, for example, A. Preisman, *Graphical constructions for vacuum tube circuits*, McGraw-Hill Publishing Co., New York, 1943.

The point E is projected over to F directly above C and D, and FD represents then the new value of current $i_1 + \Delta i_1$, at the end of the time interval Δt . Another small time interval can now be chosen, preferably equal to the previous one, so that $L/\Delta t$ remains at the same angle to the e -axis as before. We suppose that in this new time interval, e changes from $e_1 + \Delta e_1$ to $e_1 + \Delta e_1 + \Delta e_2$. Letting OG represent the new value of voltage, we project F over to H directly above G. Through H we draw HK parallel to CE, intersecting the terminal characteristic for r in K. Then KL represents the new increment of current Δi_2 , EL the additional voltage drop across r , and LH the new voltage drop across L . It is evident that Eq. (8) is once again satisfied. It is also evident that IG represents $(i_1 + \Delta i_1 + \Delta i_2)$, the new value of current at the end of the second time interval.

Points B, F, and I represent three points on the *overall terminal characteristic* for L and r in series for the given function $e(t)$. If $e(t)$ is a periodic voltage, the overall terminal characteristic will spiral around counter-clockwise and ultimately form a closed curve, the steady-state solution for the given circuit and given function $e(t)$. The initial open branches of this spiral represent the transient solution. If r is a linear resistance so that its terminal characteristic is a straight line instead of the curve shown in Fig. 11, the closed loop will be an ellipse inclined to both axes; if on the other hand r is nonlinear, the closed loop will be some form of distorted ellipse depending upon the nonlinearity of r . It can be shown from the graphical construction that the tangents to the closed loop at the points where it intersects the terminal characteristic for r are parallel to the e axis and hence perpendicular to the i axis.

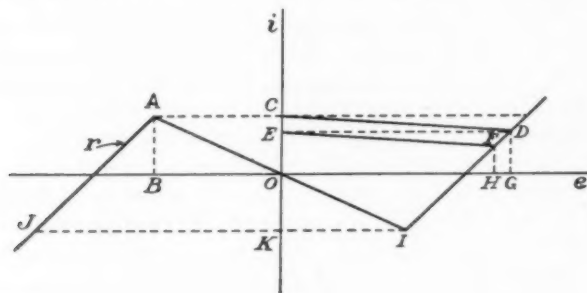


FIG. 12.

7. Relaxation oscillator. Similar methods can be developed for r in series with a condenser C , and for LCr circuits, and for parallel as well as series arrangements. Owing to lack of space these will not be treated here.² An interesting case is that of a nonlinear resistance having a suitable negative branch, in series with a pure inductance. For graphical purposes the simplest form for the terminal characteristic of r is possibly that of three intersecting straight lines, as shown in Fig. 12. Such a characteristic may be approximated by a tube having positive feedback, by a dynatron, etc. Usually a d.c. polarizing voltage is required, but this merely represents a translation of the axes and does not materially change the construction or results as obtained in Fig. 12, in which the impressed voltage is assumed to be zero.

² Cf. Preisman, loc. cit., p. 109.

Suppose that the initial conditions at $t=0$ are that $e=0$, and that $i=AB$, the peak current for the left-hand portion of r . Then C will be the starting point, where $CO=AB$. Through C the finite operator $L/\Delta t$ is drawn corresponding to a time interval Δt . If Δt is sufficiently small, $L/\Delta t$ will be practically a horizontal line through C . In Fig. 12, $L/\Delta t$ has been drawn with a finite tilt to clarify the construction, and is represented by CD . This finite operator curve intersects the terminal characteristic for r in D , as shown.

The current therefore decreases from CO to DG . Point D is projected over to the i axis as point E . From E , EF is drawn parallel to CD under the assumption that the second time interval is equal to the first. The current now decreases from DG to FH . Point F can now be projected over to the i axis and the process repeated. It is clear from the figure that the intersections will proceed down the right-hand branch of r to I , hop over from I to J , directly opposite I , then proceed from J up to A , hop over to D , and repeat the first set of intersections. As Δt approaches zero, the finite operator curve approaches a horizontal position, $DG \cong CO=AB$, and the points of intersection become more and more closely spaced so that they form essentially all the points of ID and JA .

The overall terminal characteristic is by definition all the points between C and K in that the overall impressed voltage has been assumed zero, so that the points must lie along the i axis, and the current range is from C to K . However, a more significant terminal characteristic in this case is the relationship between the current and the voltage across either circuit element. The voltage across the inductance, for example, is equal and opposite to that across r when taken in a circuital direction, since the algebraic sum of the two must equal the impressed voltage, which is zero.

According to this definition, the terminal characteristic is represented by such points as D , F , etc.; in this case, it is lines DI , IJ , JA , and AD , traversed in the order given. This means that for the circuit given, the terminal characteristic is very simply given by a quadrilateral involving the two positive resistance portions of the terminal characteristic for r contained between their peak values A and I .

The time required to traverse these portions depends upon the relaxation time for L in series with the incremental resistance of r for each portion, under the proper initial conditions. The time required to traverse the horizontal portions AD and IJ is infinitesimal, and is independent of the shape of the negative resistance portion AI provided it has no maxima or minima exceeding or less than A and I , respectively. The device operates continuously as an oscillator with a period of oscillation determined by the two relaxation times.

Similar conclusions can be drawn for shapes of r other than three straight lines. For example, r can have the form of a cubic parabola. This case has been treated analytically by Van der Pol.³ However, he started with an LCr parallel circuit or double-energy condition. For such a circuit the terminal characteristic is a closed curve or loop that exceeds the above quadrilateral in size. As C approaches zero, the loop shrinks and appears to have as its limit the above quadrilateral. However, the analytical method required that some capacity be present even in this limit, relaxation case, and it has been suggested that in a practical circuit there would always be some residual stray capacitance present.

³ B. Van der Pol, *The nonlinear theory of electric oscillations*, Proc. I.R.E., 22, 1051-1086 (1934).

There are other graphical methods for handling the double-energy case, notably that by Liénard⁴ and another by Kirschstein.⁵ Unfortunately, these constructions become indeterminate in nature as C approaches zero, so that although the relaxation condition is suggested by them, it cannot be conclusively shown to be the limit form.

The construction given here starts out with merely L and r , and requires no C for its argument. It appears to give the limit case directly and presents no indeterminate considerations. It has seemed to the author that the necessity for requiring a capacity to be present, no matter how small, was an unnecessary restriction, and that the argument advanced that any practical circuit would have some capacity, appeared to be rather irrelevant, since the notion of a circuit is in itself an idealization of what is really a field problem. In treating an electrical problem as a circuit problem one assumes that the circuit elements are ideal inductances or capacitances or resistances and develops the various theorems on this basis.

Similar results can be obtained for a capacitance in series with a nonlinear resistance having an S-shaped terminal characteristic provided that it is turned through a right angle from that shown in Fig. 12, i.e., provided that it is a single-valued function of the current rather than of the voltage. A familiar example is the neon tube relaxation oscillator employed to generate a saw-tooth voltage. It is also possible to develop a graphical construction employing the finite operator method for an LCr circuit, and in this case L or C may be permitted to approach zero, depending upon the position of the S-shaped characteristic for r , without the construction becoming indeterminate. For example, the construction reduces to the form given in connection with Fig. 12 if C is made to approach zero and r has the terminal characteristic shown in the figure.

8. Conclusions. This concludes the discussion on some graphical methods for solving nonlinear electrical circuits. Simple series circuits involving resistance elements only, are very simply solved by finding the intersections of their terminal characteristics. This can then be extended to more complicated resistances in which the current is a function of two voltages, as in the case of a triode tube.

The next circuit considered is that of the ideal balanced amplifier having perfectly matched tubes and feeding the load resistance through an ideal transformer. Here the coupling of the two tubes through this ideal transformer requires a special construction involving the sliding of a rule at a fixed angle along the tube characteristics. The wave shape of the output and of the mid-branch currents is then discussed, and it is shown that owing to the symmetry of the circuit the former can contain only odd harmonics; and the latter, even harmonics.

Finally, a simple case of a reactive circuit involving a nonlinear resistance in series with an inductance is treated. Here the concept of a finite operator curve corresponding to $L/\Delta t$ is developed and this curve is employed to solve the circuit. Similar methods are available for capacitive circuits and for double-energy circuits involving both L and C . The method is applied to a suitable negative resistance in series with an inductance, and it is shown in a direct manner that this circuit can produce relaxation oscillations.

⁴ A. Liénard, *Étude des oscillations entretenues*, Rev. Gen. Elec. 23, 901-946 (1928). See also P. LeCorbeiller, *The non-linear theory of the maintenance of oscillations*, Journal IEE (London) 79, 361-378 (1936).

⁵ F. Kirschstein, *Über ein Verfahren zur graphischen Behandlung elektrischer Schwingungsvorgänge*, Arch. Elek. 24, 731 (1930).

PRESSURE FLOW OF A TURBULENT FLUID BETWEEN TWO INFINITE PARALLEL PLANES*

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1. Introduction. The solution of the Navier-Stokes differential equations for the steady laminar flow through a channel or a circular pipe is well known for its mathematical simplicity. The reason for this simplicity is that for such flows Prandtl's boundary layer equations hold rigorously for the entire region of the fluid. In other words the boundary layer extends up to the center of the channel, whereas in the case of the flow around a solid obstacle there is only a thin layer of viscous fluid attached to the surface of the obstacle.

The steady turbulent flow through a channel or a circular pipe is more complicated in the sense that all the equations of mean motion and the equations of double and triple correlation previously developed^{1,2} have to be utilized to account for the mean velocity distribution in the entire region of the channel, and that they can not be further simplified by physical arguments as proposed, for example, by the boundary layer theory. However, if we examine the algebraic equation that represents the mean velocity distribution across the channel, we notice that it has functional behaviour similar to that of the formula for the mean velocity distribution within a turbulent boundary layer.³ In other words, the turbulent flow in a channel bears some resemblance to the corresponding laminar flow on the whole, though its detailed structure is much more complicated as will be seen soon.

In what follows we shall first determine the mean velocity distribution based upon the equation of mean motion and the equations of double correlation, by giving the triple correlations their values in the middle of the channel. This procedure leads to good results in the theory of the spread of turbulent jets and wakes (references at the end of II), but in the present case it only agrees with the experiment in the central portion of the channel, while it fails when the side is approached. We shall also see that the mean squares of the three components of the velocity fluctuation agree qualitatively with observation in the corresponding region.

The second determination given below for the mean velocity distribution utilizes equations of mean motion and both the equations of double and triple correlation by neglecting terms involving quadruple correlations. It will be shown that the triple correlations which represent the transport of turbulent energy play a particularly important role in the vicinity of the wall of the channel, and therefore can not be dispensed with for a better representation of the mean velocity distribution.

From this second determination we shall find that neglect of terms involving quadruple correlations is justifiable as a first approximation. In other words the equations

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¹ P. Y. Chou, *Chin. Journ. of Phys.* **4**, 1-33 (1940). This paper will be referred to hereafter as I.

² P. Y. Chou, *On velocity correlations and the solutions of the equations of turbulent fluctuation*, *Quart. of Appl. Math.* **3**, 38-54 (1945). This paper will be referred to as II.

³ N. Hu, *The turbulent flow along a semi-infinite plate* (unpublished).

of mean motion and of the double and triple correlations are sufficient in treating turbulent flow problems even though there is a wall present. Hence the mathematical procedure is comparatively simple in another sense that the building of equations satisfied by higher order correlations can be dropped up to the present degree of approximation.

The first determination reveals that the variation of the mean squares of the turbulent fluctuation is slower than the corresponding variation of the mean velocity distribution across the channel, which agrees qualitatively with experiment. In view of the fact that measurements of the mean squares of the components of turbulent fluctuation have not been reported systematically in the literature for the flow under consideration, we shall omit the quantitative comparison of the theory with the experimental data now available on these quantities.

In the second determination the mean velocity distribution will be calculated by assuming constant mean squares of turbulent velocity components across the channel. This is justifiable due to the slower variation of these functions across the channel, and furthermore the mean velocity distribution remains practically unchanged in the major portion of the channel—with the exception of the immediate neighborhood of the wall—when the constant values assumed for these functions are different from each other. This procedure of assigning constant values to the mean squares of the velocity fluctuations and then calculating the mean velocity distribution can be considered as the initial step in a method of iteration which will be explained in §3 below in greater detail.

In the final section we shall indicate the uncertainties connected with the correlation integrals pointed out before (II, §8). They are probably not important for the mean velocity distribution, because they involve possibly the mean squares of the turbulent fluctuation which are taken to be constant for the present calculation. These uncertainties could be removed, if we had better experimental information on the variation of the turbulent level across the channel and on the velocity correlation between two distinct points in flows such as the one examined here. In other words the present theory is perhaps sufficient so far as the mean velocity distribution is concerned, and it points out the possibilities for future investigations in turbulence along both experimental and theoretical lines.

2. Mean velocity distribution based upon the solution of the equations of mean motion and of double correlation. As before (I, §4) we take the positive x -axis ($x=x^1$) as the direction of mean motion of the fluid, the y -axis ($y=x^2$) perpendicular to the two parallel planes forming the channel, and the z -axis ($z=x^3$) parallel to these planes. The plane in mid-channel is chosen as the xz -plane. From the equations of mean motion we have

$$\tau_{12}/\rho = -U_r^2 \sigma - \nu dU/dy, \quad (2.1)$$

where

$$-\partial \bar{p}/\rho \partial x = U_r^2/d, \quad \sigma = y/d. \quad (2.2)$$

The quantity $2d$ represents the width of the channel and U_r is the so-called friction velocity.

Equation (2.1) defines the shearing stress τ_{12} in terms of y and dU/dy . Except in the immediate neighborhood of the wall the viscous stress is small, so τ_{12} is a linear

function of y . On the wall $-\nu dU/dy$ should be equal to U_τ^2 , and τ_{12} should tend toward zero as a limit.

The components τ_{23} and τ_{31} vanish due to symmetry, as pointed out before (I, §4).

From now on for simplicity we shall neglect the action of viscosity in the form of laminar stress in all the equations of motion. A physical condition mentioned previously is that all average values over time are functions of y only. Furthermore in the present special case in accordance with the definitions in II, Eqs. (5.3), those components of the slowly varying tensors a_{mik} and b_{ik} which have a single appearance of the index 3 either in i or k must be both identically zero. The vanishing of these functions is based upon the same argument as in the case of τ_{23} . The non-vanishing equations of the second order correlation (II, (8.2)) then become

$$-\frac{2}{\rho} \tau_{12} \frac{dU}{dy} + \frac{d}{dy} \overline{w_1^2 w_2} = -a_{2111} \frac{dU}{dy} - b_{11} + \frac{2\nu}{3\lambda^2} (k-5)q^2 - \frac{2\nu k}{\lambda^2} \overline{w_1^2}, \quad (2.3)$$

$$-\frac{1}{\rho} \tau_{22} \frac{dU}{dy} + \frac{d}{dy} \overline{w_1 w_2^2} = -a_{2112} \frac{dU}{dy} - b_{12} - \frac{2\nu k}{\lambda^2} \overline{w_1 w_2}, \quad (2.4)$$

$$\frac{d}{dy} \overline{w_2^3} = -a_{2122} \frac{dU}{dy} - b_{22} + \frac{2\nu}{3\lambda^2} (k-5)q^2 - \frac{2\nu k}{\lambda^2} \overline{w_2^2}, \quad (2.5)$$

$$\frac{d}{dy} \overline{w_2 w_3^2} = -a_{2133} \frac{dU}{dy} - b_{33} + \frac{2\nu}{3\lambda^2} (k-5)q^2 - \frac{2\nu k}{\lambda^2} \overline{w_3^2}. \quad (2.6)$$

These are obtained by giving i and k the sets of values (1, 1), (1, 2), (2, 2), (3, 3), respectively.

In the above four equations q is the root-mean-square of velocity fluctuation defined by

$$q^2 = \overline{w_j w_j}, \quad (2.7)$$

and k is a constant. The slowly varying tensors a_{mik} and b_{ik} should obey the divergence relations [II, (5.4)]

$$a_{2111} + a_{2122} + a_{2133} = 0, \quad b_{11} + b_{22} + b_{33} = 0. \quad (2.8)$$

The equation of vorticity decay [II, (7.11)] satisfied by Taylor's scale of micro-turbulence λ becomes in the present case

$$-14G\overline{w_1 w_2} dU/dy - 70Fq^3/3\sqrt{3} = -2\nu E q^2/3\lambda^2, \quad (2.9)$$

where E , F and G are regarded as constants.

The constant G in (2.9) is probably not important, for in the center of the channel the term involving G in (2.9) is zero due to the vanishing of dU/dy there, and in the immediate neighborhood of the wall $\overline{w_1 w_2}$ vanishes although $-dU/dy$ is large. Hence for simplicity we choose G to be zero. In fact the presence of G would only change our results slightly, as will be seen. The physical meaning of neglecting G in (2.9) is that the term that represents the creation of vorticity by deformation of the mean motion is negligible when compared with those due to transport and decay.

If G were set equal to zero, Eq. (2.9) yields

$$\lambda q/\nu = \sqrt{3}E/35F = R_0, \quad (2.10)$$

where R_0 is a constant number.

Now we shall substitute the values of the triple correlations at the center of the channel into Eqs. (2.3)–(2.6) according to their odd and even properties as functions of y . We can set

$$\overline{w_1^2 w_2} = U_\tau^3 \alpha_1 \sigma, \quad \overline{w_2^3} = U_\tau^3 \alpha_2 \sigma, \quad \overline{w_2 w_3^2} = U_\tau^3 \alpha_3 \sigma, \quad \overline{w_1 w_2^3} = U_\tau^3 \alpha_4, \quad (2.11)$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are four dimensionless constants. Although the factor U_τ^3 in (2.11) is introduced for dimensional reasons, it is possible that these four constants are all independent of the Reynolds number of the mean flow.

If we substitute from (2.10) and (2.11) into (2.3), (2.5) and (2.6), add the three together and take into account the conservation relations (2.8), we find that the mean square of the velocity fluctuation q^2 satisfies the relation

$$\frac{q^4}{U_\tau^4} = \frac{R_0^2}{10R_\tau} \left[\alpha - \frac{2\sigma}{U_\tau} \frac{dU}{d\sigma} \right], \quad (2.12)$$

where

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3, \quad R_\tau = U_\tau d / \nu; \quad (2.13)$$

R_τ is called the friction Reynolds number.

Relation (2.12) is very significant, for it tells us that for large values of $-dU/d\sigma$, q^2 varies as the square root of $-\sigma dU/d\sigma$. Within a large portion of the channel, $dU/d\sigma$ is proportional to σ , so the dependence of q^2 upon σ is fairly linear. This linear dependence has been observed to some extent for $\overline{w_1^2}$ by Wattendorf and Kuethe⁴ and by Wattendorf and Baker,⁵ and has been anticipated in the light of von Kármán's law of similarity.

If G were different from zero, the above procedure would lead to

$$\frac{q^2}{U_\tau^2} = R_0 \left[c - 2(1 + 35G) \frac{\sigma}{U_\tau} \frac{dU}{d\sigma} \right] / \sqrt{10R_\tau} \left(c - \frac{2\sigma}{U_\tau} \frac{dU}{d\sigma} \right)^{1/2} \quad (2.14)$$

which has a functional behaviour similar to that of (2.12) for large values of $-dU/d\sigma$.

It is apparent that Eqs. (2.3), (2.5) and (2.6) will determine w_1, w_2 and w_3 separately. Here we encounter the uncertainty pointed out in II, §8 that the slowly varying functions a_{nmik} and b_{ik} may contain powers or even more complicated functions of q as factors, and the existing experimental data do not provide enough evidence for a quantitative comparison with these theoretical formulae. If, for the sake of mathematical convenience, we assume b_{11}, b_{22} and b_{33} to be constant, and a_{2111}, a_{2122} and a_{2133} , which are odd functions of σ , to be proportional to σ , then $\overline{w_1^2}, \overline{w_2^2}$ and $\overline{w_3^2}$ will behave very much like q^2 , that is, when σ is near zero, $\overline{w_1^2}, \overline{w_2^2}$ and $\overline{w_3^2}$ are constants, and when σ is large and near unity, they are all proportional to the square root of $-\sigma dU/d\sigma$.

The equation for determining the mean velocity distribution is given by (2.4) which can now be written, on account of (2.10) and the condition that $w_1 w_2$ is constant, in the form

⁴ F. L. Wattendorf and A. M. Kuethe, Physics 5, 153–164 (1934).

⁵ Th. von Kármán, Proceedings of the Fifth International Congress for Applied Mechanics (Cambridge, Mass. 1938), p. 349.

$$\frac{1}{U_\tau^2} (\overline{w_2^2} + a_{2112}) \frac{1}{U_\tau} \frac{dU}{d\sigma} = - \frac{2kR_0 q^2}{R_0^2 U_\tau^2} \sigma; \quad (2.15)$$

here we have set the odd function b_{12} equal to zero for simplicity.

As pointed out before the dependence of a_{2112} upon q in the above equation is also not known. If $\overline{w_2^2}$ and q^2 are both regarded as constants, the mean velocity distribution according to (2.15) is parabolic, which agrees with experimental data fairly well for the range of σ from 0 to 0.8, and fails near the walls of the channel. This parabolic law of velocity distribution has been suggested by Stanton⁶ in his measurements of flows through a circular pipe of which the channel is a limiting case.

It has been calculated, though details will not be shown here, that this parabolic distribution of the mean velocity for constant q^2 and $\overline{w_2}$ is not essentially changed if we solve for $\overline{w_1^2}$, $\overline{w_2^2}$, $\overline{w_3^2}$ and $dU/d\sigma$ simultaneously under the further assumption that both a_{mjk}^n and b_{ijk} are equal to zero. This condition is equivalent to the vanishing of $(\overline{\omega_i w_k} + \overline{\omega_k w_i})/\rho$, which means that the shearing interaction between the pressure gradient and velocity fluctuations is zero; it has been used in jets and wakes, as mentioned before. The reason why the velocity distribution is parabolic even for this more rigorous treatment is not difficult to see without going into detailed calculations. For in the neighborhood of $\sigma=0$, both $\overline{w_2^2}$ and q^2 are constants, so $dU/d\sigma$ is proportional to σ . When the values of σ are near unity, both q^2 and $\overline{w_2^2}$ are proportional to the square root of $-\sigma dU/d\sigma$, and hence mutually proportional; consequently Eq. (2.15) again shows that $dU/d\sigma$ is proportional to σ even in the vicinity of the channel wall. We should anticipate, by the same argument, that similar simultaneous solutions for $\overline{w_1^2}$, $\overline{w_2^2}$, $\overline{w_3^2}$ and $dU/d\sigma$ would hold true even under the more general condition that b_{11} , b_{22} and b_{33} be constants and a_{2111} , a_{2122} and a_{2133} be proportional to σ as mentioned previously.

In §4 below we shall compare the numerical values of R_0 , R , and α of (2.12) with available measurements.

3. Equations of triple correlation and the mean velocity distribution. The non-vanishing equations of the triple correlation [II, (8.3)] for the present problem can be written in the form

$$3\overline{w_1^2 w_2} \frac{dU}{dy} + \frac{d}{dy} \overline{w_1^3 w_2} = -b_{21111} \frac{dU}{dy} - c_{111} + \frac{3}{\rho^2} \tau_{11} \frac{d\tau_{12}}{dy}, \quad (3.1)$$

$$2\overline{w_1 w_2^2} \frac{dU}{dy} + \frac{d}{dy} \overline{w_1^2 w_2^2} = -b_{21112} \frac{dU}{dy} - c_{112} + \frac{1}{\rho^2} \left(2\tau_{12} \frac{d\tau_{12}}{dy} + \tau_{11} \frac{d\tau_{22}}{dy} \right), \quad (3.2)$$

$$\overline{w_2^3} \frac{dU}{dy} + \frac{d}{dy} \overline{w_1 w_2^3} = -b_{21122} \frac{dU}{dy} - c_{122} + \frac{1}{\rho^2} \left(\tau_{22} \frac{d\tau_{12}}{dy} + 2\tau_{12} \frac{d\tau_{22}}{dy} \right), \quad (3.3)$$

$$\overline{w_2 w_3^2} \frac{dU}{dy} + \frac{d}{dy} \overline{w_1 w_2 w_3^2} = -b_{21133} \frac{dU}{dy} - c_{133} + \frac{1}{\rho^2} \tau_{33} \frac{d\tau_{12}}{dy}, \quad (3.4)$$

$$\frac{d}{dy} \overline{w_2^4} = -b_{21222} \frac{dU}{dy} - c_{222} + \frac{3}{\rho^2} \tau_{22} \frac{d\tau_{22}}{dy}, \quad (3.5)$$

⁶ T. E. Stanton, Proc. Roy. Soc. London. (A) 85, 366-376 (1911).

$$\frac{d}{dy} \overline{w_2^2 w_3^2} = -b_{21233} \frac{dU}{dy} - c_{233} + \frac{1}{\rho^2} \tau_{33} \frac{d\tau_{22}}{dy}. \quad (3.6)$$

These are obtained by giving i , k and l the sets of values (1, 1, 1), (1, 1, 2), (1, 2, 2), (1, 3, 3), (2, 2, 2), (2, 3, 3), respectively. The other component tensor equations in which the index 3 appears an odd number of times, namely, (1, 1, 3), (1, 2, 3), (2, 2, 3) and (3, 3, 3), are all identically zero, as are the corresponding equations of the second order correlation.

From the discussions in the previous section it is apparent that Eqs. (2.3), (2.5) and (2.6) are used to determine the mean squares of the fluctuation components, and elimination of the triple correlations between these three equations and (3.1), (3.3) and (3.4) respectively will give a more accurate determination of them. As pointed out before, existing experimental data are not accurate enough to give a quantitative comparison with the theory, and we shall not go into these detailed calculations here. Furthermore Eqs. (3.5) and (3.6) lead to quantities which are still beyond experimental proof; discussions of them will also be omitted for the present.

The elimination of the triple correlation $\overline{w_1 w_2^2}$ between (2.4) and (3.2) leads to the equation for the mean velocity distribution. Before writing down this equation we shall introduce a few more simplifications. In the first place the even function b_{21112} , which may depend upon q as mentioned previously (II, §8), is assumed to be a constant; likewise the odd function c_{112} is taken to be proportional to y and is put in the form,

$$c_{112} = 2cU_\tau^4/d. \quad (3.7)$$

It is also possible that the dimensionless number c may be a function of q and therefore an implicit function of the coordinate y .

The quadruple correlation $\overline{w_1^2 w_2^2}$ in (3.2) is of the same order of magnitude as $\overline{(w_1 w_2)^2}$ and $\overline{w_1^2 w_2^2}$. As a first approximation we shall neglect all of these terms and it will be shown afterwards in §4 that this approximation is justifiable. In short, (3.2) defines the triple correlation $\overline{w_1 w_2}$ approximately by

$$\overline{w_1 w_2^2} = -\frac{1}{2} b_{21112} - cU_\tau^4 \sigma \left/ \frac{dU}{d\sigma} \right. \quad (3.8)$$

Utilizing the above relation and (2.10) which is derived from the equation of vorticity decay, we find, after setting b_{12} in (2.4) equal to zero for mathematical convenience, that

$$\frac{a}{U_\tau} \frac{dU}{d\sigma} + c \frac{d}{d\sigma} \frac{U_\tau \sigma}{\frac{dU}{d\sigma}} = b\sigma, \quad (3.9)$$

where

$$a = -(\overline{w_2^2} + a_{2112})/U_\tau^2, \quad b = 2kR_0 q^2/R_0^2 U_\tau^2. \quad (3.10)$$

The physical meaning of the three terms in the above equation is as follows: the term in a represents the creation of turbulent energy partly due to deformation of the mean flow (I, §3(a)) and on account of a_{2112} partly contributed by the shear due to the pressure fluctuation $(\overline{\omega_k w_k} + \overline{\omega_k w_i})/\rho$; the term in b denotes the decay of turbulence; the term in c denotes the transport of turbulent energy.

The definitions of a and b in (3.10) show that they depend upon q^2 and $\overline{w_2^2}$, and are therefore functions of σ . Since q^2 as well as $\overline{w_2^2}$ as shown in the previous section varies much more slowly than $dU/d\sigma$ itself across the channel, we shall regard them as constants as the initial step to solve for $dU/d\sigma$. This initial process can also be regarded as the first step in the method of iteration in solving the present problem of turbulent flow. The second step will be to substitute this expression obtained for the mean velocity into (2.3), (2.5) and (2.6) after eliminating the triple correlations by means of (3.1), (3.3) and (3.4), and to solve for $\overline{w_1^2}$, $\overline{w_2^2}$ and $\overline{w_3^2}$. As the third step in this procedure, we utilize these values of the mean squares of the fluctuation components and solve (3.9) again for $dU/d\sigma$, and see whether the new result agrees with the solution obtained in the first step. Obviously this procedure of obtaining alternately the mean velocity and mean squares of the turbulent fluctuation can be extended indefinitely.

In the present paper we shall not follow this refined method of approach; instead we shall solve (3.9) by assigning constant values to a , b and c , or to q^2 and $\overline{w_2^2}$, and compare the different solutions by varying these constants. The result will be that except in the immediate neighborhood of the wall of the channel, the different mean velocity distributions according to (3.9) for the different sets of a , b and c respectively agree well with each other and with experiment, showing that the variation of the mean squares of the turbulent fluctuation across the channel does not influence the mean velocity distribution very much.

The solution of (3.9) with constant a , b and c is

$$\frac{1}{2}\sigma^2 = aU/bU_\tau + cA_1e^{bU/cU_\tau} + A_2, \quad (3.11)$$

where A_1 and A_2 are two constants of integration.

If A_1 in (3.11) is zero, then (3.10) gives a parabolic law of velocity distribution and a must be negative, since b according to its definition in (3.10) is positive. The presence of the term in A_1 gives the so-called "logarithmic law" of velocity distribution which holds true especially in the neighborhood of the wall of the channel. Hence the product cA_1 can not be zero. It is apparent that this exponential term in U/U_τ is due to the presence of the triple correlation in Eq. (3.9).

The boundary conditions used to determine the constants cA_1 , A_2 and the ratio a/b are:

$$\text{when } \sigma = 0, \quad U = U_c; \quad \text{when } \sigma = 1, \quad U = 0, \quad -dU/U_\tau d\sigma = \infty. \quad (3.12)$$

The value U_c denotes the maximum velocity of the flow in mid-channel. We have chosen the derivative $-dU/U_\tau d\sigma$ on the wall of the channel to be infinite. In fact, it should be R_τ which is a fairly large number. Since we are interested in the mean velocity distribution within the channel proper, substituting infinity for the friction Reynolds number R_τ gives a good approximation.

The boundary conditions (3.12) render (3.11) into the following final form,

$$(e^{\kappa U_c/U_\tau} - \kappa U_c/U_\tau - 1)\sigma^2 = -\kappa(U_c - U)/U_\tau + e^{\kappa U_c/U_\tau}[1 - e^{-\kappa(U_c - U)/U_\tau}], \quad (3.13)$$

where $\kappa = b/c$ and $b/a = 2[\exp(\kappa U_c/U_\tau) - \kappa U_c/U_\tau - 1]/\kappa$.

Equation (3.13) expresses the mean velocity defect $(U_c - U)/U_\tau$ as a function of σ with two parameters κ and U_c/U_τ . The presence of these two constants may appear at the first sight to contradict the experimental velocity defect law formulated by

von Kármán,⁷ according to which $(U_c - U)/U_\tau$ should be independent of the Reynolds number of the mean flow which, in turn, is a function of the ratio U_c/U_τ . A close examination of the experimental data shows, however, that this discrepancy is not serious. In the first place von Kármán's velocity defect law can only hold true in the central portion of the channel and there is a dependence of the velocity defect upon the Reynolds number in the vicinity of the channel wall. It has been shown that for flows in circular pipes U_c/U_τ increases from about 19 to 33 when the friction Reynolds number $2aU_\tau/\nu$ changes from $\sqrt{10^6}$ to 10^6 , $2a$ being the diameter of the pipe.⁸ In the second place even formula (3.13), which does indicate the dependence of $(U_c - U)/U_\tau$ upon U_c/U_τ , can only account for the mean velocity distribution in the interior of the channel for a given set of constants a , b and c in (3.11), and these constants have to take another set of values in the turbulent boundary layer on the wall, although the same functional behaviour of (3.11) still prevails within the layer.⁸ This point will be discussed in greater detail in the following section.

The quantity U_c/U_τ in (3.13) is given by experiment; then the constant κ is fixed, for instance, by passing the theoretical curve through the experimental point at $\sigma = 0.7$. In view of the variation of the ratio U_c/U_τ with the Reynolds number of the mean flow, we shall choose a few different values of κ and calculate the mean velocity

TABLE 1. $(U_c - U)/U_\tau$

	(1)	(2)	(3)	(4)	(5)
$\sigma \backslash \kappa$	Obs.	-0.1	0.0	+0.1	0.2151
0.00	0.00	0.00	0.00	0.00	0.00
0.10	0.16	0.06	0.05	0.05	0.05
0.20	0.38	0.23	0.22	0.21	0.19
0.30	0.66	0.52	0.50	0.48	0.44
0.40	1.10	0.95	0.91	0.88	0.81
0.50	1.64	1.50	1.47	1.42	1.34
0.60	2.33	2.22	2.19	2.15	2.08
0.70	3.13	3.13	3.13	3.13	3.13
0.80	4.28	4.31	4.38	4.50	4.75
0.90	6.30	5.92	6.17	6.60	7.72
0.93	—	6.58	6.91	7.51	9.30
0.96	8.81	7.39	7.88	8.74	11.84
0.98	—	8.12	8.76	9.93	15.01
0.99	—	8.64	9.40	10.82	18.21
1.00	—	9.86	10.94	13.07	∞

distribution. This will lead to different values of U_c/U_τ . But we shall see that for all these cases the mean velocity distributions agree with each other and with experiment within the channel proper.

Let us calculate the mean velocity distribution for the values of κ equal to -0.1, 0, 0.1 and 0.2151, and determine the corresponding values of U_c/U_τ by passing the theoretical curves through the experimental point at $\sigma = 0.7$. The equations that determine $(U_c - U)/U_\tau$ for $\kappa = -0.1$ and 0.1 are given respectively by

⁷ Th. von Kármán, Proceedings of the Fourth International Congress for Applied Mechanics, Cambridge 1934, p. 70.

⁸ S. Goldstein, *Modern developments in fluid dynamics*, vol. 2, The Clarendon Press, Oxford, 1938, p. 338.

$$\sigma^2 = 0.2785(U_c - U)/U_\tau - 1.039[e^{0.1(U_c - U)/U_\tau} - 1], \quad (3.14)$$

$$\sigma^2 = -1.5870(U_c - U)/U_\tau + 2.662[1 - e^{-0.1(U_c - U)/U_\tau}]. \quad (3.15)$$

For the case $\kappa=0$, we can get a limiting equation by letting κ approach zero in (3.13), or we can solve (3.9) directly by setting b equal to zero. The latter procedure leads to

$$(U_c - U)/U_\tau = 10.94[1 - (1 - \sigma^2)^{1/2}], \quad (3.16)$$

where 10.94 is the value of $(c/a)^{1/2}$.

The case when $\kappa=0.2151$ is represented by

$$\sigma^2 = 1 - e^{-0.2151(U_c - U)/U_\tau}. \quad (3.17)$$

This is the solution of (3.11) with a set equal to zero; the numerical value 0.2151 stands for the ratio b/c .

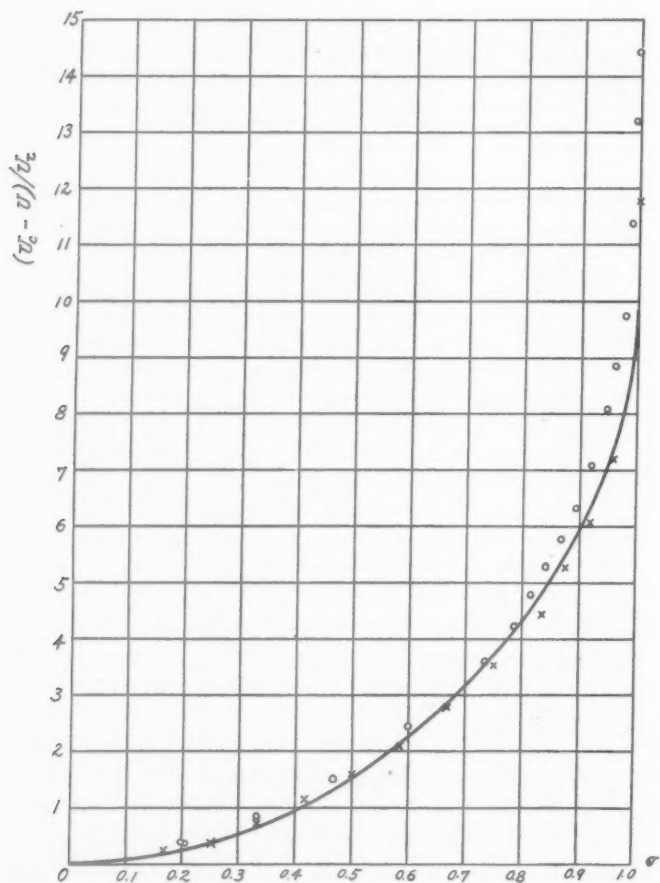


FIG. 1. Velocity distributions in a channel.

The experimental values of $(U_c - U)/U_r$ which are taken from a paper by Goldstein⁹ are given in column (1) of Table 1; the corresponding theoretical values according to (3.14), (3.16), (3.15) and (3.17) are tabulated in columns (2), (3), (4) and (5), respectively.

From this table we see that as the value of κ increases from -0.1 to $+0.2151$, U_c/U_r changes from 9.86 to ∞ . Hence 0.2151 is the maximum limiting value of κ for constant a, b, c in (3.9). Equation (3.17) shows in this limiting case that the mean velocity distribution in the whole channel is "logarithmic."

In order to avoid confusion, only the solution (3.14) for $\kappa = -0.1$ is plotted in Fig. 1. The circles represent Dönch's measurements¹⁰ found for $U_m d/\nu$ equal to 8.7×10^4 , U_m being the average value of U over a cross section of the channel. The crosses reproduce Nikuradse's results¹¹ for $U_m d/\nu$ equal to 3.3×10^4 . It is seen that apart from the immediate neighborhood of the channel wall, agreement between theory and experiment is satisfactory.

4. Relation between the present theory and some known experimental data. From the foregoing calculations we see that we can subject to experimental test not only the mean velocity defect distribution $(U_c - U)/U_r$ and the mean squares of the fluctuation components, but also the relation (2.10) between λ and q and the relation (3.8) which approximates the triple correlation $\overline{w_1 w_2^2}$. Let us examine relations (2.10) and (2.12) first.

The experimental data used by Taylor in his statistical theory¹² are, in c.g.s. units; $U_c = 114$ cm/sec, $U_r = 5.39$ cm/sec, $\rho = 0.00123$, $\nu = 0.14$, $d = 12.3$ cm. Since according to (2.10) $R_0 = \lambda q/\nu$ is a constant, we can compute R_0 from the values of λ and q in the center of the channel. In Taylor's table λ^2 in mid-channel is equal to 2.9 cm², so λ is 1.7 cm. The mean magnitude of the velocity fluctuation q at this point is roughly $1.2 U_r$ [cf. (4.2) below]. These values then give

$$R_0 = 78.5, \quad R_r = 474. \quad (4.1)$$

We have shown above in §2 that $\overline{w_1^2}$ behaves very much like q^2 . The experimental values of $\overline{w_1^2}/U_r^2$ across the channel, determined by Wattendorf and Baker⁵ for the flow with Reynolds number $109,000$, can be represented by

$$\overline{w_1^2}/U_r^2 = 0.412(1 + 27.2\sigma^2)^{1/2}. \quad (4.2)$$

As far as the order of magnitude is concerned, $\overline{w_1^2}$ can be put equal to $q^2/3$. By comparing (4.2) with (2.12), we find that

$$R_0 \sqrt{\alpha}/\sqrt{10R_r} \sim 3 \times 0.4 = 1.2. \quad (4.3)$$

If we use a parabolic representation of the mean velocity distribution that goes through the experimental point at $\sigma = 0.7$, we have

$$-2\sigma dU/U_r d\sigma = 25.6\sigma^2. \quad (4.4)$$

Then α in (2.10) becomes $25.6/27.2 = 0.94$. Putting this value of α in (4.3), we obtain $R_0^2/R_r \sim 10 \times 1.4/0.94 = 15$, while from (4.1) we find that $R_0^2/R_r \sim 13$.

⁹ S. Goldstein, Proc. Roy. Soc. London (A) **159**, 473-496 (1937).

¹⁰ F. Dönch, Forschungsarbeiten des Ver. Deutsch. Ing. no. **282** (1926).

¹¹ J. Nikuradse, Forschungsarbeiten des Ver. Deutsch. Ing. no. **289** (1929).

¹² G. I. Taylor, Proc. Roy. Soc. London (A) **151**, 456 (1935).

This shows the order of agreement between the two sets of values obtained from two entirely different experimental sources. It must be pointed out, however, that the number $R_\tau = 474$ for the experimental value of λ used in the above calculation may be too low for the flow to be in a fully developed turbulent state.

Equation (2.10) also shows the dependence of R_0 upon the quantities E and F which occur in the definitions of the double and triple correlation functions between two distinct points [II, (6.8), (6.11)]. The measurement of these functions separately will give another check on the value of the number R_0 discussed above.

We next study the values of the three constants a , b and c in Eqs. (3.9) and their physical significance.

(1) $\kappa = -0.1$. According to (3.10) b must be positive, so c in this case must be negative. The definition of c/a from (3.13) and (3.14) gives

$$c/a = 20/0.2785 = 72. \quad (4.5)$$

Hence a must be also negative. If a_{2112} in (3.10) were zero, a becomes of the order of 0.4 and c is equal to 29. From the definition of c in (3.7), we find that

$$c_{112} = 58U_\tau^4/d, \quad (4.6)$$

which is 29 times larger than $2\tau_{12}d\tau_{12}/\rho^2dy$, a term of the same order of magnitude as the one in the quadruple correlation $\overline{dw_1^2w_2^2}/dy$. Hence all these terms are negligible as a first approximation.

After the value of c is known, the triple correlation function $\overline{w_1w_2^2}$ is determined uniquely according to (3.8); the constant term $\frac{1}{2}b_{2112}$ is fixed by the value of $\overline{w_1w_2^2}$ at $\sigma=0$.

By means of $b = -0.1c$, the definition of b in (3.10) and the values of R_τ/R_0^2 and q^2/U_τ^2 given before, we find the numerical value of k to be of the order of 16.

(2) $\kappa=0$, $(c/a)^{1/2}=10.94$, $c/a=120$. This gives results similar to those in case (1) and consequently the terms in the quadruple correlations in (3.2) are still negligible. In this case c and a must be negative as in the foregoing example. The meaning of $b=0$ is that the term due to the decay of turbulence in (3.9) is negligible when compared with the other two.

(3) $\kappa=0.1$. In this case c should be positive. Then the definition of c/a from (3.13) and (3.15) gives $c/a=20/1.5870 \sim 13$, and a must be also positive. The condition that a be greater than zero changes the picture a great deal, for then a_{2112} in (3.10) is negative and its magnitude is greater than $\overline{w_2^2}$. Nevertheless the terms in the quadruple correlations are still negligible, if the absolute value of a_{2112} is, say, a few multiples of $\overline{w_2^2}$.

(4) $\kappa=0.2151$, $a=0$. Here we have $b/c=0.2151$. Putting this value into the definition of b from (3.10), we have $c \sim 10 \times k \times 1.2/15$, which is about 12, if k is of the order of 15. This gives c_{112} of (3.7) equal to $24U_\tau^4\sigma/d$, a quantity still about 10 times greater than the terms involving the quadruple correlations in (3.2). The physical significance of $a=0$ means that in (3.9) the term due to deformation is small when compared with the terms due to transport of and the decay of the turbulent energies. Obviously from the equations of double correlation (2.3)–(2.6) the magnitude of k

can be determined by measurement of the mean squares of the velocity fluctuation components.

From the four alternative cases discussed above we see that although the measurement of the mean velocity distribution alone will not single out which one is the correct theoretical mean velocity distribution, measurements of the variations of the higher order correlation functions across the channel will decide this question. For example, experiment on the triple correlation $\overline{w_1 w_2^2}$ in (3.8) will decide whether c is negative or positive, and the theoretical pattern for the mean velocity distribution can thus be determined.

From another angle the above four special cases can also be considered to represent the mean velocity distribution in four parts of the channel. In the central portion, we have negative a and negative c [cf. (3.10), (3.8)]. Since both c and a_{2112} can be functions of the coordinate σ , their values may change at the various points of the channel. It is possible that a may eventually become positive as σ increases near the wall, while c which was negative in mid-channel, increases to zero and finally becomes positive on the wall of the channel.

According to its definition in (3.10), b is positive and is a monotonically increasing function of the distance from the center; likewise κ increases monotonically with σ , if c has already become positive. This increasing property of κ as the wall is approached is substantiated experimentally. In Hu's theory of the turbulent flow along a semi-infinite plate,⁸ the mean velocity distribution in the turbulent boundary layer can be represented by an equation analogous to (3.11), and the value of b/c is equal to 0.4 instead of 0.1 as in case (3). Hence our present solution for mean motion only covers the channel proper; if the boundary layer on the channel wall is approached, the solution should be replaced by Hu's result. In fact it is well-known experimentally that the turbulent boundary layer on the wall covers the region $30 < R_\tau(1 - \sigma) < 250$. In Dönch's measurement¹⁰ cited above, R_τ is equal to 3630, so the range $0.931 < \sigma < 0.992$ represents approximately the turbulent boundary layer on the wall and we should expect formula (3.13) to fail in this region. A rigorous theory to explain the mean velocity distribution for the entire channel including the boundary layer might not be impossible according to present indications, but the actual mathematical manipulation involved would be much more complicated than that in the present treatment.

5. Conclusion. Based upon the foregoing analysis in the cases of the four values of κ for the motion of a turbulent fluid through a channel, we may conclude that the velocity defect distribution $(U_c - U)/U_\tau$, which is practically independent of the Reynolds number of the mean flow within the channel proper according to von Kármán, is also independent of the magnitudes of the turbulent fluctuation when the flow has reached the steady turbulent state. The question as to whether the above conclusion can be generalized to state that the double and triple correlation distributions across the channel when expressed in terms of the frictional velocity U_τ , namely, the ratios $\overline{w_i w_j}/U_\tau^2$ and $\overline{w_i w_j w_k}/U_\tau^3$, are also independent of the Reynolds number of the mean flow and of the correlations of still higher orders remains to be seen theoretically as well as experimentally. In any event, the friction velocity U_τ probably plays an important role for turbulent flow problems involving the presence of a wall, as in the present problem.

DIFFUSION IN TURBULENT FLOW BETWEEN PARALLEL PLANES*

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1. Introduction. The equation of diffusion in a turbulent fluid

$$\frac{\partial^2 \chi}{\partial z^2} + \frac{1-2p}{z} \frac{\partial \chi}{\partial z} = \frac{\partial \chi}{\partial x} \quad (1)$$

(χ stands for temperature, vapour concentration, or whatever property is being studied, x is measured in the direction of mean flow and z in the perpendicular direction, and p is a constant determined by the degree of turbulence of the fluid) was introduced by O. G. Sutton¹ and extensively studied by W. G. L. Sutton,² who considered a number of cases of diffusion in the semi-infinite region $z > 0$. It has been shown by Pasquill³ that for the semi-infinite region the theory is in good agreement with experiments, both on evaporation and on heat transfer.

In this note a number of results for symmetrical flow in the finite region $0 < z < 2l$ will be given; it is assumed that $2l$ is small enough for the power law velocity profile to hold up to the centre of the region. Such cases are of some practical interest, and may provide an indication of the behaviour to be expected in the much more difficult problem of heat transfer in a circular pipe. Also they are interesting generalizations of known solutions of the equation of conduction of heat in the rod $0 < z < 2l$, with constant temperature, or flow of heat, at its ends.

The method used will be that of the Laplace transformation. W. G. L. Sutton (*loc. cit.*) remarks that if $p = 1/2$, equation (1) reduces to the equation of linear flow of heat, and he gives a treatment of (1) which is a generalization of Goursat's treatment of the equation of conduction of heat. It is well known that the Laplace transformation method is particularly well suited to the solution of specific problems in conduction of heat, and that its advantage increases as the complexity of the problem increases. This suggests that the method may have the same advantages when applied to (1), and, in fact, this proves to be the case. All the results of W. G. L. Sutton's paper can be obtained more shortly in this way, and explicit expressions for the solutions for more complicated boundary conditions, composite regions, etc., can also be derived.

In Section 2 the standard problem of evaporation in the semi-infinite region is solved as an illustration of the method, and for comparison with later results. In

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¹ O. G. Sutton, *Wind structure and evaporation in a turbulent atmosphere*, Proc. Roy. Soc. (A), **146**, 701 (1934).

² W. G. L. Sutton, *On the equation of diffusion in a turbulent medium*, Proc. Roy. Soc. (A), **182**, 48 (1943). The notation used here is that of this paper, except that the symbol E is introduced in (4). The variables x and z in (1) are dimensionless quantities defined in Sutton's paper.

³ F. Pasquill, *Evaporation from a plane, free-liquid surface into a turbulent air stream*, Proc. Roy. Soc. (A), **182**, 75 (1943).

Section 3 two other results for the semi-infinite region are given for completeness. In Sections 4-6 the most interesting cases of symmetrical flow in the region $0 < z < 2l$ are studied. The solutions given here are formal only, but in all cases they may be made rigorous by the verification process described elsewhere.⁴

Equation (1) has to be solved in the region $x > 0$, and in a prescribed region of z , with boundary conditions in x

$$\chi \rightarrow \chi^{(0)}(z), \text{ as } x \rightarrow +0, \quad (2)$$

$$\chi \text{ finite, as } x \rightarrow \infty. \quad (3)$$

In all the problems considered below $\chi^{(0)}(z)$ will be zero, that is the temperature or vapour concentration in the fluid is zero in the plane $x=0$.

There are also boundary conditions in z , which will be expressed either in terms of χ , or of

$$E = -Bz^{1-2p} \frac{\partial \chi}{\partial z}. \quad (4)$$

This quantity E is the local rate of diffusion across the plane $z=\text{const.}$, and B is a known constant (defined by Sutton, *loc. cit.*) involving the fluid and its degree of turbulence.

The constant p in (1) is restricted in Sutton's theory by the inequality $0 < p < 1/3$, and we assume here $0 < p < 1$.

With the substitution

$$\chi = z^p \Omega, \quad (5)$$

(1) becomes

$$\frac{\partial^2 \Omega}{\partial z^2} + \frac{1}{z} \frac{\partial \Omega}{\partial z} - \frac{p^2}{z^2} \Omega - \frac{\partial \Omega}{\partial x} = 0. \quad (6)$$

Introducing the Laplace transform of Ω with respect to x , namely

$$\Omega^* = \int_0^\infty e^{-sx} \Omega dx, \quad (7)$$

we obtain from (6) the subsidiary equation⁵ for Ω^* ,

$$\frac{d^2 \Omega^*}{dz^2} + \frac{1}{z} \frac{d\Omega^*}{dz} - \left(s + \frac{p^2}{z^2}\right) \Omega^* = -z^{-p} \chi^{(0)}(z). \quad (8)$$

2. The semi-infinite region $z > 0$. Boundary conditions: $\chi^{(0)}(z) = 0$, $z > 0$. $\chi = \chi_0$, constant,⁶ when $z = 0$, $x > 0$. χ finite, as $z \rightarrow \infty$, $x > 0$. Here (8) becomes

$$\frac{d^2 \Omega^*}{dz^2} + \frac{1}{z} \frac{d\Omega^*}{dz} - \left(s + \frac{p^2}{z^2}\right) \Omega^* = 0, \quad z > 0, \quad (9)$$

⁴ H. S. Carslaw and J. C. Jaeger, *Operational methods in applied mathematics*, Oxford, 1941, §58, and J. C. Jaeger, *Radial heat flow in a circular cylinder with a general boundary condition*, Proc. Roy. Soc. N.S.W., 75, 130-139 (1942).

⁵ For the procedure see, e.g., Carslaw and Jaeger, *loc. cit.*

⁶ For shortness, boundary conditions will usually be written in this way; it is implied, of course, that $\chi \rightarrow \chi_0$ as $z \rightarrow +0$ for fixed $x > 0$.

to be solved with

$$\chi^* = z^p \Omega^* \rightarrow \chi_0/s, \quad \text{as } z \rightarrow +0, \quad (10)$$

and

$$\chi^* \text{ finite as } z \rightarrow \infty. \quad (11)$$

The solution of (9) which satisfies (11) is $K_p(zs^{1/2})$, and since

$$z^p K_p(zs^{1/2}) \rightarrow 2^{p-1} \Gamma(p) s^{-1/2} \quad \text{as } z \rightarrow +0, \quad (12)$$

it follows that the solution of (9), (10), and (11) is

$$\chi^* = \chi_0 \frac{z^p 2^{1-p}}{\Gamma(p)} s^{1/2-p} K_p(zs^{1/2}). \quad (13)$$

Now it is known that⁷

$$s^{1/2-p} K_p(zs^{1/2}) \text{ is the Laplace transform of } z^{-p} 2^{p-1} \int_{z^2/4x}^{\infty} e^{-u} u^{p-1} du. \quad (14)$$

Thus the required solution is

$$\chi = \chi_0 \frac{1}{\Gamma(p)} \int_{z^2/4x}^{\infty} e^{-u} u^{p-1} du. \quad (15)$$

This is the result given by Pasquill (*loc. cit.*, (9)).

It follows from (15), or directly from the transform E^* of E , that

$$E \rightarrow \frac{B\chi_0 2^{1-2p} x^{-p}}{\Gamma(p)}, \quad \text{as } z \rightarrow +0, \quad (16)$$

and

$$\int_0^x E dx \rightarrow \frac{B\chi_0 2^{1-2p} x^{1-p}}{(1-p)\Gamma(p)}, \quad \text{as } z \rightarrow +0. \quad (17)$$

3. Two other results for the semi-infinite region $z > 0$. The results to be derived here are both for the case $\chi^{(0)}(z) = 0$, and $\chi(z)$ finite as $z \rightarrow \infty$.

If the boundary condition at $z=0$ is: $E \rightarrow E_0$, constant, as $z \rightarrow +0$, the solution is

$$\chi = \frac{E_0 z^{2p}}{2B\Gamma(1-p)} \int_{z^2/4x}^{\infty} e^{-u} u^{-p-1} du. \quad (18)$$

Also

$$\chi \rightarrow \frac{E_0 2^{2p-1} x^p}{Bp\Gamma(1-p)}, \quad \text{as } z \rightarrow +0. \quad (19)$$

This is proved exactly as in Section 2, using (14).

⁷ S. Goldstein, *Operational representations of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function*, Proc. London Math. Soc. (2) **34**, 104 (1932), (15) and (24). Alternatively the result can be obtained by the use of the inversion theorem for the Laplace transformation, subsequently deforming the line integral into the contour $(-\infty, 0+)$; cf. Carslaw and Jaeger, *loc. cit.*, §39. The same remark applies to the derivation of (18) and (21) below.

If the boundary condition at $z=0$ is

$$h\chi - z^{1-2p} \frac{\partial \chi}{\partial z} = h\chi_0, \quad (20)$$

where h and χ_0 are constants, the solution is

$$\chi = \chi_0 - \frac{\chi_0 z^p}{2^{p-1}\Gamma(p)} \int_0^\infty e^{-zu^2} u^{p-1} \frac{J_p(uz) + \alpha u^{2p} J_{-p}(uz)}{1 + 2\alpha u^{2p} \cos p\pi + \alpha^2 u^{4p}} du, \quad (21)$$

where $\alpha = \Gamma(1-p)2^{1-2p}/h\Gamma(p)$.

To prove (21) the inversion theorem for the Laplace transformation, (24) below, has to be used, and the line integral must be deformed into $(-\infty, 0+)$.

The result (15) was derived for a constant value of χ on the boundary $z=0$. The solution for the case in which χ is a prescribed function of x on $z=0$ can be obtained from (15) by Duhamel's theorem. The same remark applies to the cases of Sections 4 and 6. Correspondingly, the solutions of the problems of Sections 3, 5 with E a prescribed function of x on $z=0$ can be obtained in the same way.

4. The region $0 < z < l$. $\chi^{(0)}(z) = 0$. $\chi = \chi_0$, constant, when $z=0$, $x > 0$. $E=0$, when $z=l$, $x > 0$. This corresponds to the region $0 < z < 2l$ with flow symmetrical about $z=l$, and with $\chi = \chi_0$ on $z=0$ and $z=2l$, for $x > 0$. Thus, for example, it gives the solution of the problem of heat transfer from the parallel planes $z=0$ and $z=2l$, both maintained at constant temperature χ_0 , and with symmetrical flow between them.

Here we have to solve (9) with boundary conditions (10) and

$$E^* = 0, \quad \text{when } z = l. \quad (22)$$

By (12) the solution of (9) which satisfies (10) is

$$\chi^* = \frac{\chi_0 z^p 2^{1-p}}{\Gamma(p)} s^{1-p-1} K_p(zs^{\frac{1}{2}}) + A z^p I_p(zs^{\frac{1}{2}}).$$

The unknown A is found by substituting in (22), and we have finally

$$\chi^* = \frac{\chi_0 z^p 2^{1-p} s^{1-p-1} [K_p(zs^{\frac{1}{2}}) I_{p-1}(ls^{\frac{1}{2}}) + I_p(zs^{\frac{1}{2}}) K_{p-1}(ls^{\frac{1}{2}})]}{\Gamma(p) I_{p-1}(ls^{\frac{1}{2}})}. \quad (23)$$

χ is found from (23) by using the inversion theorem for the Laplace transformation [cf. Carslaw and Jaeger, *loc. cit.*]

$$\chi = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sz} \chi^*(s) ds \quad (24)$$

$$= \frac{\chi_0 z^p 2^{1-p}}{2\pi i \Gamma(p)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sz} s^{1-p-1} [K_p(zs^{\frac{1}{2}}) I_{p-1}(ls^{\frac{1}{2}}) + I_p(zs^{\frac{1}{2}}) K_{p-1}(ls^{\frac{1}{2}})] ds}{I_{p-1}(ls^{\frac{1}{2}})}, \quad (25)$$

where $\gamma > 0$.

The integrand of (25) is a single valued function⁸ of s . It has a simple pole at $s=0$ of residue

$$2^{p-1} z^{-p} \Gamma(p), \quad (26)$$

⁸ G. N. Watson, *Theory of Bessel functions*, Cambridge University Press, Cambridge 1922, §3.71, (17) and (18)

and simple poles at $s = -\alpha_r^2/l^2$, where $\pm\alpha_r$, $r = 1, 2, \dots$, are the zeros (all real and simple⁹) of

$$J_{p-1}(\alpha) = 0. \quad (27)$$

It is easy to show [cf. Jaeger, *loc. cit.*] that the line integral in (25) is equal to $2\pi i$ times the sum of the residues at the poles of its integrand. Evaluating these we get finally

$$\chi = \chi_0 - \frac{\chi_0 z^p}{2^{p-2} l^p \Gamma(p)} \sum_{r=1}^{\infty} \frac{\alpha_r^{p-2} e^{-\alpha_r^2 z/l^2} J_p(z\alpha_r/l)}{J_p^2(\alpha_r)}. \quad (28)$$

The most interesting quantity is the value of E as $z \rightarrow 0$. Either from (28), or directly by calculation of its transform, this is found to be

$$E \rightarrow \frac{B\chi_0 \Gamma(1-p)}{\Gamma(p) 2^{2p-2} l^{2p}} \sum_{r=1}^{\infty} \frac{e^{-\alpha_r^2 z/l^2} J_{1-p}(\alpha_r)}{\alpha_r^{1-2p} J_p(\alpha_r)}, \quad \text{as } z \rightarrow +0. \quad (29)$$

Also, as $z \rightarrow +0$,

$$\int_0^z E dx \rightarrow \frac{B\chi_0 l^{2-2p}}{2(1-p)} \phi_p^{(1)}(x/l^2), \quad (30)$$

where

$$\phi_p^{(1)}(x/l^2) = 1 - \frac{\Gamma(2-p) 2^{3-2p}}{\Gamma(p)} \sum_{r=1}^{\infty} \frac{e^{-\alpha_r^2 x/l^2} J_{1-p}(\alpha_r)}{\alpha_r^{3-2p} J_p(\alpha_r)}. \quad (31)$$

For small values of x/l^2 the value of $\int_0^x E dx$ given by (30) reduces to the value (17) for the semi-infinite region, and

$$\phi_p^{(1)}(x/l^2) \sim \frac{2^{2-2p} (x/l^2)^{1-p}}{\Gamma(p)}. \quad (32)$$

In Fig. 1 graphs of these quantities are shown for $p = 1/9$, the value commonly found in wind tunnel experiments. Curve I shows the result (32) for the semi-infinite region, and Curve II the value of $\phi_p^{(1)}(x/l^2)$ given by (31) for values of x/l^2 for which the difference between (31) and (32) is important. For larger values of x/l^2 than those shown the exponentials in (31) are negligible.

In the case of heat transfer the quantity (30) gives the amount of heat taken up from the region 0 to x of one of the planes.

5. The region $0 < z < l$. $\chi^{(0)}(z) = 0$, $E \rightarrow E_0$, constant, as $z \rightarrow 0$, $x > 0$. $E = 0$, $z = l$, $x > 0$. This corresponds to the region $0 < z < 2l$ with flow symmetrical about $z = l$, and with constant diffusion across the planes $z = 0$ and $z = 2l$.

Here, proceeding as in Section 4, we find

$$\chi^* = \frac{E_0 z^p 2^p \{ I_{p-1}(ls^{\frac{1}{2}}) K_p(zs^{\frac{1}{2}}) + K_{p-1}(ls^{\frac{1}{2}}) I_p(zs^{\frac{1}{2}}) \}}{B \Gamma(1-p) s^{\frac{1}{2}p+1} I_{1-p}(ls^{\frac{1}{2}})}. \quad (33)$$

The most interesting quantity to evaluate in this case is the value of χ as $z \rightarrow +0$. This is found to be

$$\frac{2E_0(1-p)l^{2p}}{B} \left[\psi_p(x/l^2) + \frac{x}{l^2} \right], \quad (34)$$

⁹ G. N. Watson, *loc. cit.*, §§15.25, 15.21. For the method of calculating their values in practice see John R. Airey, *Bessel functions of small fractional order and their application to problems of elastic stability*, Phil. Mag. (6), 41, 200 (1921).

where

$$\psi_p(x/l^2) = \frac{1-p}{2p(2-p)} - \frac{2^{2p-1}\Gamma(p)}{\Gamma(2-p)} \sum_{r=1}^{\infty} \frac{e^{-x\alpha_r^2/l^2} J_{p-1}(\alpha_r)}{\alpha_r^{2p+1} J_{-p}(\alpha_r)}, \quad (35)$$

and the α_r are the roots of

$$J_{1-p}(\alpha) = 0. \quad (36)$$

For small values of x/l^2 , (34) tends to the result (19).

If $p=1/9$ it is found that the difference between (34) and (19) is less than 1% for values of x/l^2 up to 0.3, while for greater values of x/l^2 the exponentials in (35) are

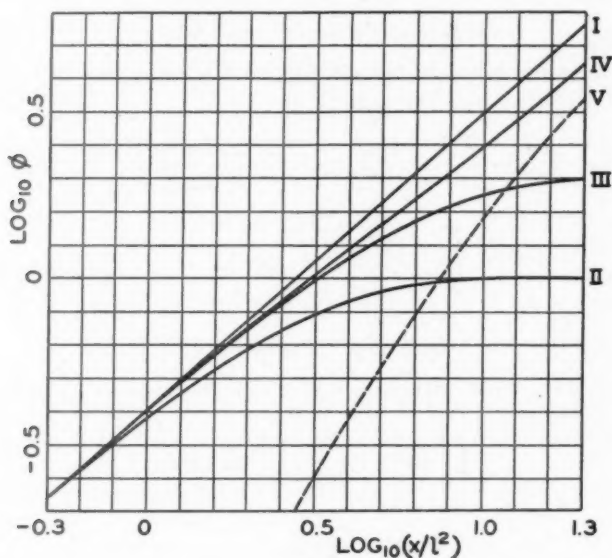


FIG. 1.

almost negligible. In the case of heat transfer, (34) gives the surface temperature of one of a pair of planes to which heat is supplied at a constant rate per unit time per unit area, and which are cooled by turbulent fluid flowing between them.

6. Two cases of symmetrical flow in the region $0 < z < 2l$. First let us consider boundary conditions $\chi^{(0)}(z) = 0$, and

$$\chi = \chi_0, \text{ constant, when } z = 0, x > 0 \quad (37)$$

$$E = 0, \text{ when } z = 2l, x > 0. \quad (38)$$

Here the regions $0 < z < l$ and $l < z < 2l$ must be treated separately. We write $\chi_1(z_1)$ and $E_1(z_1)$ for the values of χ and E in $l < z < 2l$ as functions of $z_1 = 2l - z$ in this region. The boundary conditions at the surface of separation $z = z_1 = l$ are

$$\chi = \chi_1, \quad (39) \quad E = -E_1. \quad (40)$$

A solution of (9) which satisfies (37) is

$$\chi^* = \frac{\chi_0 2^{1-p} s^{\frac{1}{2}p-1}}{\Gamma(p)} z^p K_p(zs^{\frac{1}{2}}) + A z^p I_p(zs^{\frac{1}{2}}),$$

and a solution of (9) with z replaced by z_1 which satisfies (38) is

$$\chi_1^* = C z_1^p I_{-p}(z_1 s^{\frac{1}{2}}).$$

The unknowns A and C are found by substituting in the transforms of (39) and (40), which gives

$$\chi_1^* = \frac{\chi_0 2^{1-p} z_1^p s^{\frac{1}{2}p-1} I_{-p}(z_1 s^{\frac{1}{2}})}{l \Gamma(p) [I_{-p}(ls^{\frac{1}{2}}) I_{p-1}(ls^{\frac{1}{2}}) + I_p(ls^{\frac{1}{2}}) I_{1-p}(ls^{\frac{1}{2}})]} \quad (41)$$

with a rather longer formula for χ^* . As in Section 4, χ and χ_1 are evaluated by the use of the inversion theorem and the results are

$$\chi_1 = \chi_0 - \frac{\chi_0 2^{1-p} z_1^p \Gamma(1-p)}{l^p} \sum_{r=1}^{\infty} \frac{e^{-x\alpha_r^2/l^2} \alpha_r^{p-1} J_{p-1}(\alpha_r) J_p(\alpha_r) J_{-p}(z\alpha_r/l)}{J_{p-1}^2(\alpha_r) + J_p^2(\alpha_r)}, \quad (42)$$

$$\chi = \chi_0 - \frac{\chi_0 2^{1-p} z^p}{l^p \Gamma(p)} \sum_{r=1}^{\infty} \frac{e^{-x\alpha_r^2/l^2} \alpha_r^{p-2} J_p(z\alpha_r/l)}{J_{p-1}^2(\alpha_r) + J_p^2(\alpha_r)}, \quad (43)$$

where the α_r , ($r=1, 2, \dots$), are the positive roots¹⁰ of

$$J_{-p}(\alpha) J_{p-1}(\alpha) - J_p(\alpha) J_{1-p}(\alpha) = 0. \quad (44)$$

As in Section 4 the most interesting quantity is the value of $\int_0^x E dx$ as $z \rightarrow 0$. This is found to be

$$\int_0^x E dx \rightarrow \frac{B \chi_0 l^{2-2p}}{2(1-p)} \phi_p^{(2)}(x/l^2), \quad (45)$$

where

$$\phi_p^{(2)}(x/l^2) = 2 - \frac{2^{3-2p}(1-p)}{[\Gamma(p)]^2} \sum_{r=1}^{\infty} \frac{e^{-x\alpha_r^2/l^2}}{\alpha_r^{4-2p} \{J_p^2(\alpha_r) + J_{p-1}^2(\alpha_r)\}}. \quad (46)$$

For small values of (x/l^2) , (46) behaves like (32). Its value for $p=1/9$ and for values of (x/l^2) for which the difference from (32) is important is shown in Curve III of Fig. 1; for larger values of x/l^2 the exponentials in (46) are negligible.

The result (45) gives the evaporation from the region 0 to x of the plane $z=0$ if there is no flow over the plane $z=2l$.

Finally we consider the case in which the boundary conditions are $\chi^{(0)}(z)=0$, and

$$\chi = \chi_0, \quad \text{constant, } z=0, x>0, \quad (47)$$

and

$$\chi = 0, \quad \text{when } z=2l, x>0. \quad (48)$$

Here, proceeding as before and writing χ_1 for the value of χ in $l < z < 2l$, E_1 for the value of E in this region, and $z_1=2l-z$, we find

¹⁰ In the case $k=1/2$ when (1) becomes the equation of linear flow of heat, (44) becomes $\cos 2\alpha=0$ and similarly (51) and (52) become $\sin 2\alpha=0$.

$$\chi = \chi_0 \left(1 - \frac{z^{2p}}{2l^{2p}} \right) - \frac{\chi_0 2^{1-p} z^p}{l^p \Gamma(p)} \sum_{r=1}^{\infty} \left\{ \frac{e^{-\alpha_r^2 z/l^2} J_p(z\alpha_r/l)}{\alpha_r^{2-p} J_{p-1}^2(\alpha_r)} + \frac{e^{-\beta_r^2 z/l^2} J_p(z\beta_r/l)}{\beta_r^{2-p} J_{p-1}^2(\beta_r)} \right\}, \quad (49)$$

$$\chi_1 = \frac{1}{2} \chi_0 \left(\frac{z_1}{l} \right)^{2p} + \frac{\chi_0 2^{1-p} z_1^p}{l^p \Gamma(p)} \sum_{r=1}^{\infty} \left\{ \frac{e^{-\alpha_r^2 z_1/l^2} J_p(z_1\alpha_r/l)}{\alpha_r^{2-p} J_{p-1}^2(\alpha_r)} - \frac{e^{-\beta_r^2 z_1/l^2} J_p(z_1\beta_r/l)}{\beta_r^{2-p} J_{p-1}^2(\beta_r)} \right\}, \quad (50)$$

where the α_r are the positive roots of

$$J_p(\alpha) = 0, \quad (51)$$

and the β_r are the positive roots of

$$J_{p-1}(\beta) = 0. \quad (52)$$

This problem is that of heat transfer between the plane $z=0$ at constant temperature χ_0 , and the plane $z=2l$ at zero temperature, by turbulent fluid flowing between. The quantity of heat taken up from the region 0 to x of the plane $z=0$ is determined by

$$\lim_{z \rightarrow +0} \int_0^x E dx = \frac{B\chi_0 l^{2-2p}}{2(1-p)} \phi_p^{(3)}(x/l^2), \quad (53)$$

where

$$\begin{aligned} \phi_p^{(3)}(x/l^2) &= \frac{2p(1-p)x}{l^2} + \frac{1+p+2p^2}{2(1+p)} \\ &\quad - \frac{2^{3-2p}(1-p)}{[\Gamma(p)]^2} \sum_{r=1}^{\infty} \left\{ \frac{e^{-x\alpha_r^2/l^2}}{\alpha_r^{4-2p} J_{p-1}^2(\alpha_r)} + \frac{e^{-x\beta_r^2/l^2}}{\beta_r^{4-2p} J_{p-1}^2(\beta_r)} \right\} \end{aligned} \quad (54)$$

and the α_r and β_r are defined by (51) and (52). For small values of (x/l^2) , $\phi_p^{(3)}(x/l^2)$ behaves like (32). For larger values its behaviour for the case $p=1/9$ is shown in Fig. 1, Curve IV, and for still larger values the exponentials in (54) are negligible.

The quantity of heat taken up by the region 0 to x of the plane $z=2l$ is determined by

$$- \lim_{z_1 \rightarrow +0} \int_0^x E_1 dx = \frac{B\chi_0 l^{2-2p}}{2(1-p)} \phi_p^{(4)}(x/l^2), \quad (55)$$

where

$$\begin{aligned} \phi_p^{(4)}(x/l^2) &= \frac{2p(1-p)x}{l^2} - \frac{(1+2p)(1-p)}{2(1+p)} \\ &\quad - \frac{2^{3-2p}(1-p)}{[\Gamma(p)]^2} \sum_{r=1}^{\infty} \left\{ \frac{e^{-x\alpha_r^2/l^2}}{\alpha_r^{4-2p} J_{p-1}^2(\alpha_r)} - \frac{e^{-x\beta_r^2/l^2}}{\beta_r^{4-2p} J_{p-1}^2(\beta_r)} \right\}. \end{aligned} \quad (56)$$

A portion of the curve of $\phi_p^{(4)}(x/l^2)$ for $p=1/9$ is shown in Fig. 1, Curve V; for larger values of x/l^2 the exponentials in (56) are negligible, and for smaller values than those shown in the figure $\phi_p^{(4)}$ continues to decrease rapidly.

ON THE STABILITY OF TWO-DIMENSIONAL PARALLEL FLOWS

PART II.—STABILITY IN AN INVISCID FLUID*

BY

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7. General considerations. The criteria of Rayleigh and Tollmien. At the end of Part I, we have shown that the study of the stability problem in an inviscid fluid gives valuable information provided it is kept in mind that we are actually dealing with the limiting case where the Reynolds number becomes indefinitely large. The study of the stability of two-dimensional parallel flows in an inviscid fluid is usually regarded as being quite complete, through the work of Rayleigh and Tollmien. Their results show that instability depends very much upon the occurrence of a point of inflection in the velocity profile. However, it seems that physical interpretations of such general results are not well known. Such an interpretation will be given in §§9, 10 of this part. There are also several points in the mathematical theory which require further development and clarification. These will be brought out for further consideration in §§7, 8.

We now proceed to make a critical survey of some aspects of the stability problem in an inviscid fluid. First, let us summarize the conclusions obtained by Rayleigh and Tollmien. These can be conveniently described as the necessary and the sufficient conditions for the existence of a disturbance, self-excited, neutral, or damped.

1) *Necessary conditions for the existence of a disturbance.*

a) If the flow possesses a self-excited or neutral mode of disturbance with finite wave length, the velocity profile has a flex at some point $y=y_s$, where $y_1 < y_s < y_2$. Furthermore, in the case of a *neutral* disturbance, the phase velocity must be $c=w(y_s)$.

b) If the flow possesses a *damped* mode of disturbance, no immediate conclusion can be drawn.

2) *Sufficient conditions for the existence of a disturbance.* So far, the sufficient conditions are known only for symmetrical and for boundary-layer velocity distributions. The results may be stated as follows.

a) There is always the neutral disturbance given by $c=0$, $\alpha=0$, $\phi(y)=w(y)$.

b) If $w''(y_s)=0$, for $y_1 < y_s < y_2$, there is a neutral disturbance with $c=w(y_s)$; furthermore, if $w'''(y_s) \neq 0$, self-excited disturbances also exist.

Discussion. The condition $w'''(y_s) \neq 0$ involved in (2) (b) will be shown to be actually unnecessary, by an improved method of proof to be discussed in the next section. The statement in (1) (b) regarding damped disturbances differs from the original conclusion of Rayleigh and Tollmien. Indeed, in the work of Lord Rayleigh, the solution is taken to be valid all along the real axis. Hence, in accordance with the discussion of §5, Part I, such considerations do not include damped disturbances. However, Rayleigh and Tollmien did not distinguish between an amplified disturbance and a damped disturbance, because they regarded them as complex conjugates. As pointed out in §5, this is not permissible. In fact, if we accept the original conclusions

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of Rayleigh and Tollmien, a profile without a flex could not execute any kind of disturbance. This can hardly be reconciled with our intuition regarding the state of affairs in a real fluid at infinitely large Reynolds numbers.¹ According to the present interpretation, only damped solutions can exist. Such a conclusion is also borne out by the investigations for a viscous fluid.² It is to be noted that the neutral and the self-excited disturbances, existing under the condition $w''(y_s)=0$, are free from the effect of viscosity inside the fluid, because the neutral solution is also regular at $y=y_s$, where $w=c$. Hence, we may conclude that *disturbances essentially free from the effect of viscosity inside a fluid can exist only for velocity distributions with a flex.*

The results of Rayleigh and of Tollmien discussed above tend to give the impression that the occurrence of a flex in the profile is the decisive factor in the determination of instability not only in the case of an inviscid fluid, but also in the case of a viscous fluid.³ However, the investigation in Part III will show that this is by no means the case. When instability first occurs, as one increases the Reynolds number, viscous forces still play a dominant role, and the main characteristics of the behavior of the fluid with respect to a disturbance do not depend upon the occurrence of a flex in the velocity curve. Indeed, it is physically improbable that a slight change of the pressure gradient in the case of a boundary layer—which may cause a change from a velocity curve without a flex to one with a flex—should cause a radical change in the essential characteristics of stability. As we shall see later, the instability of a boundary layer depends more on the outside free stream than on the occurrence of a point of inflection. It might be argued that the free stream is analogous to a point of inflection in that a vanishing curvature is involved; but even if this is admitted, we must still note that the essential features in this case are not obtained from an investigation neglecting the effect of viscosity. Indeed, from inviscid investigations, it is concluded that a boundary layer with zero or favorable pressure gradient is stable, except for the very trivial type of disturbance with infinite wave-length and zero phase velocity. The present investigation shows that *all* boundary-layer profiles can be unstable, and exhibits results in agreement with the physical suggestion just discussed.

It thus seems that *any conclusion obtained from inviscid investigations must not be taken over directly to the case of the real fluid, where the stability phenomenon is largely controlled by the effect of viscosity and not decided primarily by the occurrence of a flex in the velocity curve.*

Indeed, even when we are mainly interested in the behavior in the limiting case of infinite Reynolds numbers, the existence of a flex is not as significant as it may appear to be at first sight. The existence of neutral or amplified disturbances has so far been proved only for *symmetrical* and *boundary-layer* types of velocity profiles. This may not be true for other types of velocity profiles, e.g., when the walls are in relative motion. The following example will bring out this point. Let us consider the velocity distribution $w(y)=A+B\sin y$, $y_1 < y < y_2$, which has a flex at $y=0$ if $y_1 < 0 < y_2$. According to the above necessary conditions, the only possible neutral

¹ This is the objection of Friedrichs, loc. cit. (Ref. [5]) p. 209. (The references are listed at the end of Part I.) It must also be noted that the non-linear terms are not negligible in the case of an *ideal* fluid. We shall consistently restrict the magnitude of our disturbances so that the effect of viscosity is always more important than the effect of non-linearity.

² See figures in Ref. [27].

³ See Taylor's discussion on p. 308 of Ref. [70].

disturbance is the one with $c=A$. Then the equation of disturbance (6.21) reduces to $\phi'' + (1 - \alpha^2)\phi = 0$. It has the solution

$$\phi(y) = C \sin \{ \sqrt{1 - \alpha^2} (y - y_1) \},$$

which vanishes at $y = y_1$. If $\phi(y_2)$ is also required to vanish, we must have

$$\sqrt{1 - \alpha^2} (y_2 - y_1) = n\pi, \quad (n = \text{integer}),$$

and hence

$$\alpha^2 = 1 - [n\pi / (y_2 - y_1)]^2.$$

Thus, if $y_2 - y_1 < \pi$, there is no possible neutral disturbance; if $y_2 - y_1 = \pi$, there is the one with $\alpha = 0$; if $\pi < y_2 - y_1 < 2\pi$, there is one with $\alpha \neq 0$; in general, if $m\pi < y_2 - y_1 < (m+1)\pi$, there are m neutral disturbances with $\alpha \neq 0$. In the last case, there are also m points of inflection in the velocity profile.

It can thus be seen that the general shape of the velocity profile plays a very important role even in the limit of infinite Reynolds numbers. Indeed, it will become clear from Part III that the eigen solution with eigen values $\alpha = c = 0$ is not as trivial as it might appear at first sight, for it actually represents a limiting case with $R \rightarrow \infty$. This solution exists for symmetrical and boundary-layer profiles, but its existence is not immediately evident for other types of profiles.

In spite of all these points against the *decisive* nature of the flex, it must be admitted that its occurrence certainly makes the motion comparatively unstable. This can be expected from the original results of Rayleigh and Tollmien, and can be seen more clearly from the interpretation of the mechanism of inertia forces to be given in §§9, 10. However, these results must not be taken to indicate any decisive nature of a flex. The essential features of instability can only be obtained through consideration of the effect of viscosity.

We shall now conclude this section by making some critical discussions of Heisenberg's classification of velocity profiles and the use of broken linear profiles for the study of stability problems.

Heisenberg's classification of velocity distributions. Heisenberg attempted the case of flow between solid walls in relative motion with the condition that $\text{Re}(w - c)$ vanishes only once for $y_1 < y < y_2$ (loc. cit., p. 592). Regarding α^2 as small, he approximated the condition (6.18) by $K_1(c) = 0$ [cf. (6.26), (6.24)]. He then classified the profile into four classes: (i) those for which $K_1(c) = 0$ has a complex root; (ii) those for which $K_1(c) = 0$ has a real root; (iii) those for which the real part of $K_1(c)$ vanishes for a certain real value of c ; (iv) those for which none of the above three cases is true. Heisenberg concluded that the first class is unstable, the second generally unstable, the rest stable.

In discussing the validity of these conclusions, the following point must be borne in mind. If we can show that a certain type of disturbance exists for $\alpha^2 = 0$ and $\alpha R \rightarrow \infty$, it may also be expected to exist for sufficiently large values of αR and sufficiently small values of α^2 . However, the non-existence of a certain type of disturbance for $\alpha^2 = 0$ and $\alpha R \rightarrow \infty$ does not exclude the possibility of its existence for finite values of α^2 and αR . It appears therefore that we can only expect to conclude the *instability* of a velocity distribution by discussing the roots of $K_1(c) = 0$. Thus, apart from some flaws in Heisenberg's mathematical deductions, only the first two classes can have any decisive significance.

If $K_1(c)$ has a root with a *positive* imaginary part, the motion is unstable. If $K_1(c)$ has a *real* root, Heisenberg shows that the motion would be unstable when the effect of viscosity is considered. This will be studied more fully in a generalized form in §11. However, if $K_1(c)$ has a root with a *negative* imaginary part, we cannot conclude the instability of the flow by taking the complex conjugate of $K_1(c)=0$ (as Heisenberg did). For if

$$\int_C dy(w-c)^{-2} = 0,$$

then (cf. Fig. 5)

$$\int_{C'} dy(w-c)^{-2} = -2\pi i R_0,$$

where R_0 is the residue of $(w-c)^{-2}$ at y_0 . In fact $R_0 = -w_0''/w_0'^3$. Now $K_1(\bar{c})$ is the complex conjugate of $\int_{C'} dy(w-c)^{-2}$. Hence,

$$K_1(\bar{c}) = 2\pi i \bar{R}_0 = -2\pi i \bar{w}_0''/\bar{w}_0'^3,$$

which does not vanish unless $w_0''=0$. Hence, the equation $K_1(c)=0$ tells us nothing about the existence of the root \bar{c} or any other root with a positive imaginary part.

Thus, Heisenberg's attempt appears to be not as successful as Tollmien's later work [75], which at least brings out the characteristic properties of symmetrical and boundary-layer distributions. A complete classification of velocity distributions, however, is not yet existent.

Approximation using broken linear profiles. Some investigations of Lord Rayleigh were carried out by approximating the velocity profile with straight-line segments. With this approximation, the solutions of (6.21) can be expressed in terms of elementary functions. Lord Rayleigh also tried to verify his conclusions by considering the roots of $K_1(c)=0$, using the same approximation for the velocity. However, the results of his investigations are doubtful, because the number of roots obtained for $K_1(c)$ is equal to the number of corners chosen in the approximation. This was demonstrated by Heisenberg to be inherent in the method of approximation. The general idea is as follows. As discussed above, the stability condition (6.18) may be approximated by $K_1(c)=0$ in certain cases. Although Rayleigh's approximation may be made very close so far as the velocity distribution is concerned, the approximation to $(w-c)^{-2}$ is always bad in the neighborhood of the corners. Consequently, the integral $K_1(c)$ is not properly approximated. In fact, a *continuous* broken profile $w(y)$ does not allow itself to be continued analytically to the complex y -plane without introducing *discontinuities* (cuts). It thus appears that all results deduced from the consideration of broken profiles must be regarded with reserve. The same criticism applies to Tietjen's work with the viscous fluid. His analysis failed to give a minimum Reynolds number below which all small disturbances are damped out.

8. Rigorous proof and extension of Tollmien's result for the existence of unstable modes of oscillation. In this section, we want to give a rigorous proof of the existence of amplified solutions of (6.21) satisfying the second and the third boundary conditions of (6.22) when the velocity profile $w(y)$ has a flex at $y=y_0$, i.e.,

$$w''(y_0) = 0. \quad (8.1)$$

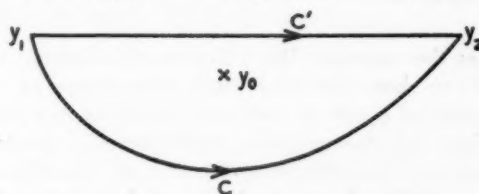


FIG. 5. Path around the critical point in the case $c_1 < 0$.

The idea of the proof is essentially the same as that used by Tollmien, but the method is improved. It has the further advantage of enabling us to extend the results to cover cases where $w'''(y_s) = 0$,—a condition which had to be excluded by Tollmien.

According to previous results, the neutral disturbance must have a phase velocity c equal to

$$c_s = w_s = w(y_s). \quad (8.2)$$

Let the corresponding value of α be denoted by α_s . The essential idea of the proof is (1) to show that there exist eigen-values of c and $\alpha > 0$ in the neighborhood of the values of c_s and α_s such that the imaginary part of c does not vanish, and then (2) to show that the imaginary part is actually positive. The first statement can be expected and can be readily established, if we can show that the left-hand sides of (6.18)–(6.20) are analytic functions $f(\alpha, c)$ of the two variables α and c in the neighborhoods of α_s and c_s . For if this is true, we can always solve $f(\alpha, c) = 0$ for c as an analytic function of α , (there may be more than one branch), by the implicit function theorem. Hence, there is at least one value of c corresponding to every *real* value of α in the neighborhood of $\alpha = \alpha_s$. Furthermore, by (8.2), this value of c , being unequal to c_s , cannot be real, and the first part of our result is established.

To prove the analyticity of $f(\alpha, c)$ seems to be a trivial problem. Nevertheless, we shall find below that it is impossible to establish it in the neighborhood of $(\alpha, c) = (0, 0)$. The chief problem in the proof is to overcome the difficulty caused by the singular point of the differential equation (6.21).

If $w - c \neq 0$, we can write (6.21) in the form

$$\phi'' - \alpha^2 \phi - \frac{w''}{w - c} \phi = 0. \quad (8.3)$$

Let us now consider a simply-connected region R of the y -plane which encloses the

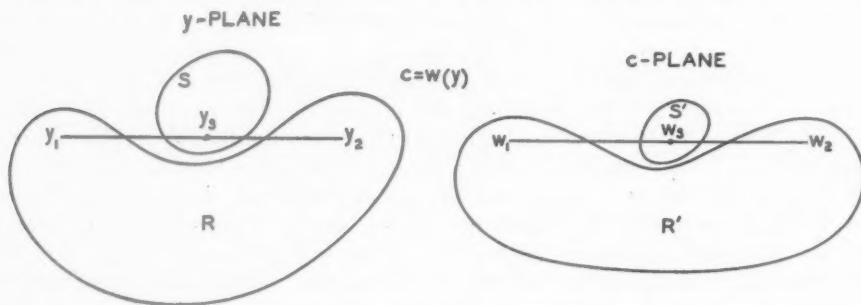


FIG. 6. The region of analyticity of the inviscid solutions.

points $y = y_1$ and $y = y_2$, but excludes the point y_s , the passage from y_1 to y_2 being taken in the *lower* half of the y -plane. We consider also a neighborhood S of y_s , mutually exclusive with the region R . Let us regard the relation

$$c = w(y) \quad (8.4)$$

as mapping the regions R and S into two regions R' and S' of the c -plane (Fig. 6). If the mapping is one-to-one, (as can be expected if $w'(y) \neq 0$ for $y_1 < y < y_2$), these

regions will also be mutually exclusive. Then, if we restrict y to R and c to S' , the coefficients of (8.3) are analytic functions of the independent variable y and the parameters α and c . Hence, a fundamental system of solutions of (8.3), which we denote by $\phi_1(y; \alpha, c)$ and $\phi_2(y; \alpha, c)$, are *analytic* functions of the *three* variables y , α , and c . We understand that y is restricted to the region R , c is restricted to the region S' , while α may be in any finite region enclosing α_s . Thus, (for example),

$$f_2(\alpha, c) \equiv \begin{vmatrix} \phi_1(y_1; \alpha, c) & \phi_2(y_1; \alpha, c) \\ \phi_1'(y_2; \alpha, c) & \phi_2'(y_2; \alpha, c) \end{vmatrix} \quad (8.5)$$

is an analytic function of the variables α and c , as we want to prove.

We note that in the neighborhood of $(\alpha, c) = (0, 0)$, the above reasoning fails. The region R (which has to enclose the point $y = y_1$) and the region S (which has to enclose the point where $w = c = 0$) cannot be taken to be mutually exclusive. In fact, $f(\alpha, c)$ presumably has a singular point at the point $\alpha = 0$ (a logarithmic branch point). We shall discuss this case a little more closely at the end of this section.

Let us proceed to show that there actually exist values of $c = c(\alpha^2)$ with a *positive* imaginary part corresponding to positive real values of α . This is necessary because the usual argument of taking complex conjugates has been shown to be invalid. For this purpose, we consider the power series

$$c = c_s + \left(\frac{dc}{d\lambda}\right)_s (\lambda - \lambda_s) + \frac{1}{2!} \left(\frac{d^2c}{d\lambda^2}\right)_s (\lambda - \lambda_s)^2 + \dots, \quad (8.6)$$

where $\lambda = \alpha^2$.⁴ Since λ is restricted to real values, the important point to be shown is that *the first of the derivatives in (8.6) for which the imaginary part does not vanish is of odd order*. Then, by taking values of λ slightly greater or smaller than λ_s , we can always make $c_i > 0$. For these values of c and α^2 , we can continue our solution $\phi(y)$ analytically so that it is given along the real axis between y_1 and y_2 , thus obtaining an inviscid solution.

Let us now consider (8.3), writing λ for α^2 . We have

$$L(\phi) \equiv \phi'' - \lambda\phi - \frac{w''}{w - c} \phi = 0. \quad (8.7)$$

Let ϕ be an eigen function with λ, c as the corresponding eigen-values. Then

$$L(\phi_\lambda) \equiv \phi_\lambda'' - \lambda^2 \phi_\lambda - \frac{w''}{w - c} \phi_\lambda = \left\{ 1 + \frac{w''}{(w - c)^2} \frac{dc}{d\lambda} \right\} \phi, \quad (8.8)$$

where

$$\phi_\lambda = \frac{\partial \phi}{\partial \lambda} + \frac{\partial \phi}{\partial c} \frac{dc}{d\lambda}.$$

We distinguish two cases: (1) the point $y = y_s$ is a simple root of $w''(y) = 0$; (2) the point $y = y_s$ is a multiple root of $w''(y) = 0$.

In the first case, $w'''(y_s) \neq 0$. In the limit $\lambda \rightarrow \lambda_s, c \rightarrow c_s$, Eqs. (8.7) and (8.8) become⁵

⁴ Since $d\lambda/d\alpha \neq 0$ at $\alpha = \alpha_s$, the correspondence between α and λ is one-to-one in the neighborhood of $\lambda_s = \alpha_s^2$.

⁵ A subscript s denotes that the parameters λ and c are put equal to λ_s and c_s respectively. A subscript λ denotes differentiation with respect to λ .

$$L_s(\phi_s) \equiv \phi_s'' - \lambda_s \phi_s - \frac{w''}{w - c_s} \phi_s = 0, \quad (8.9)$$

$$L_s(\phi_{\lambda s}) \equiv \phi_{\lambda s}' - \lambda_s \phi_{\lambda s} - \frac{w''}{w - c_s} \phi_{\lambda s} = \left\{ 1 + \frac{w''}{(w - c_s)^2} \left(\frac{dc}{d\lambda} \right)_s \right\} \phi_s. \quad (8.10)$$

From these, we deduce that

$$\phi_s L_s(\phi_{\lambda s}) - \phi_{\lambda s} L_s(\phi_s) = \frac{d}{dy} \{ \phi_s \phi_{\lambda s}' - \phi_{\lambda s} \phi_s' \} = \left\{ 1 + \frac{w''}{(w - c_s)^2} \left(\frac{dc}{d\lambda} \right)_s \right\} \phi_s^2.$$

Now, ϕ_{λ} satisfies the same boundary conditions as ϕ does, because those conditions are satisfied by ϕ for each pair of values of λ and c , and ϕ is an analytic function of them. Hence, integrating $\phi_s L_s(\phi_{\lambda s}) - \phi_{\lambda s} L_s(\phi_s)$ between the limits (y_1, y_2) , we have

$$\left(\frac{dc}{d\lambda} \right)_s \int_{y_1}^{y_2} \frac{w''}{(w - c_s)^2} \phi_s^2 dy + \int_{y_1}^{y_2} \phi_s^2 dy = 0,$$

or

$$\left(\frac{dc}{d\lambda} \right)_s = - \int_{y_1}^{y_2} \phi_s^2 dy / \int_{y_1}^{y_2} w'' \phi_s^2 (w - c_s)^{-2} dy. \quad (8.11)$$

The denominator of the above expression is equal to

$$\begin{aligned} \int_{y_1}^{y_2} (w_s''' y + \frac{1}{2} w_s^{iv} y^2 + \dots) (w_s' y + \frac{1}{2} w_s''' y^2 + \dots)^{-2} (\phi_{ss}^2 + 2\phi_{ss}\phi_{ss}' y + \dots) dy \\ = \frac{w_s'''^2}{w_s'^2} \phi_{ss}^2 \int_{y_1}^{y_2} \left\{ \frac{1}{y} + A_0 + A_1 y + \dots \right\} dy, \end{aligned}$$

where ϕ_{ss} is the value of ϕ_s at $y = y_s$, and A_0, A_1, \dots are real. Hence, the imaginary part of the above expression is $\pi \phi_{ss} w_s'''^2 / w_s'^2$. Since $\phi_s(y)$ is real and ϕ_{ss} does not vanish,⁶ we have arrived at the required result. The above argument is a rigorous formulation of Tollmien's work.

In case $w''(y)$ has a multiple root at $y = y_s$, the proof of Tollmien does not hold, but the above method can still be carried through. The restriction must be made, however, that the point y_s is a point of inflection where $w''(y)$ actually changes its sign. Then, y_s is a root of $w''(y)$ of odd multiplicity, and the first of the derivatives $w^{iv}(y_s), w^v(y_s), \dots$ which does not vanish is of *odd* order. Such a point always exists when the curvature of the velocity curve has different signs at y_1 and y_2 . If we differentiate (8.7) n times with respect to λ , we have the following equation for each value of n :

$$L(\phi_{\lambda^{(n)}}) = n\phi_{\lambda^{(n-1)}} + \sum_{r=1}^n C_r \phi_{\lambda^{(n-r)}} \frac{d^r}{d\lambda^r} \left(\frac{w''}{w - c} \right). \quad (8.12)$$

Let $w''(y)$ have the root y_s up to the multiplicity $2m+1$, $m > 0$. Then, (8.10) is regular in a neighborhood of $y = y_s$, and the value of $(dc/d\lambda)_s$ as given by (8.11) is *real*. Let us consider the boundary value problem of the differential equation (8.10), requiring $\phi_{\lambda s}$ to satisfy the same boundary conditions as ϕ_s . The solution can be ob-

⁶ Tollmien [74], p. 92.

tained from $\phi_\lambda(y)$ by making $\lambda \rightarrow \lambda_s$, and is moreover real along the real axis, by a direct consideration of (8.10).

Continuing the same argument with equations of the type (8.12) with $n=2, 3, \dots, 2m$ and $c \rightarrow c_s, \lambda \rightarrow \lambda_s$, we find that

$$(\phi_\lambda^{(2)})_s, (\phi_\lambda^{(3)})_s, \dots, (\phi_\lambda^{(2m)})_s; \left(\frac{d^2c}{d\lambda^2}\right)_s, \left(\frac{d^3c}{d\lambda^3}\right)_s, \dots, \left(\frac{d^{2m}c}{d\lambda^{2m}}\right)_s$$

are all real. Finally, for $n=2m+1$, we obtain a relation of the type

$$(2m+1)! \left(\frac{dc}{d\lambda}\right)_s^{2m+1} \int_{y_1}^{y_2} w'' \phi_s^2 (w-c)^{-(2m+2)} dy + \left(\frac{d^{2m+1}c}{d\lambda^{2m+1}}\right)_s \int_{y_1}^{y_2} \phi_s^2 dy = \text{real.} \quad (8.13)$$

Just as in the case of the equation preceding (8.11), it can be easily seen that the above integral $\int_{y_1}^{y_2} w'' \phi_s^2 (w-c)^{-(2m+2)} dy$ has the imaginary part $w_s^{(2m+3)} \phi_{ss}^2 / (2m+1)! (w'_s)^{(2m+2)}$ while the other term on the left-hand side of (8.13) is real. Thus, $(d^{2m+1}c/d\lambda^{2m+1})_s$ has a non-vanishing imaginary part. This is the result desired.

This completes the proof of the existence of amplified solutions near the neutral solution $c=c_s, \alpha=\alpha_s$ when the velocity curve has a point of inflection.

The proof of the existence of amplified solutions near the neutral solution $c=0, \alpha=0$ cannot be so easily formulated into a rigorous form. From the solutions (4.14), it is very easy to obtain the solution ϕ_I which approaches the eigen solution $\phi=w(y)$ as $c \rightarrow 0, \alpha^2 \rightarrow 0$, with $\alpha^2 = O(c)$. The solution is

$$\begin{aligned} \phi_I = & -cw'_1(w-c) \int_{y_1}^y (w-c)^{-2} dy \\ & \times \left\{ 1 + \alpha^2 \int_{y_1}^y dy (w-c)^2 \int_{y_1}^y dy (w-c)^{-2} + \dots \right\}. \end{aligned} \quad (8.14)$$

As can be easily verified from (6.21) the condition that ϕ_I be an eigen function is

$$cw'_1 + \alpha^2 \int_{y_1}^{y_2} (w-c) \phi_I dy = 0. \quad (8.15)$$

From this, it follows that

$$\left(\frac{dc}{d\lambda}\right)_0 = \frac{1}{w'_1} \int_{y_1}^{y_2} w^2 dy, \quad (8.16)$$

and that the imaginary part of $(d^2c/d\lambda^2)_0$ is $2\pi(dc/d\lambda)_0 w'_1/w_1'^2$, which is positive if there is one flex in the velocity profile ($w'_1 > 0$). However, the real part of $(d^2c/d\lambda^2)_0$ becomes logarithmically infinite, and hence the argument is not rigorous. Also, it does not seem easy to make suitable modifications and extensions in case $w''(y_1)=0$. It should be remarked that Tollmien's proof is not essentially different from the argument just given.

Similar considerations can be applied to boundary-layer profiles, and similar results can be obtained confirming and extending Tollmien's original results.

9. Physical interpretation of instability in an inviscid fluid. The fact that the instability of a two-dimensional parallel laminar flow is so closely connected with the occurrence of a point of inflection in the velocity profile demands a physical inter-

pretation. Since Eq. (6.21) is essentially the vorticity equation, we would expect $w''=0$ to indicate a maximum or minimum of the vorticity $-w'$ of the main flow. This is actually where the explanation is to be found.

Since we have neglected the effect of viscosity, a fluid element maintains its vorticity throughout the motion. From this point of view, a two-dimensional parallel flow may be regarded as the motion of a large number of vortex filaments under the action of each other. Filaments of equal vorticity are arranged in the same layer, and the whole flow is built up of a collection of such layers.

The following physical interpretation is based upon the fact that a fluid element is accelerated in such a field if it is associated with an excess or a defect of vorticity. These considerations were originally developed by von Kármán⁷ for the interpretation of the failure of the simple vorticity-transfer theory of fully developed turbulence as applied to the case of parallel Couette flow. The idea is developed in greater mathematical detail here in this section and the next. It will be noticed that the consideration is essentially two-dimensional, and hence is even more suitable here than for fully developed turbulence, where the fluctuations are essentially three-dimensional. An alternative interpretation of the results of Rayleigh and Tollmien, but still based upon vorticity considerations, will also be given to demonstrate the role of the viscous forces.

Let us imagine a disturbance of the flow such that an element E_1 of fluid of the layer L_1 is interchanged with an element E_2 of a neighboring layer L_2 . For definiteness, let us suppose that the layer L_2 has a higher vorticity than the layer L_1 in the undisturbed state. Since E_1 preserves its vorticity, it will appear to have a defect of vorticity when it is in L_2 . Similarly, E_2 appears to have an excess of vorticity.

Let us fix our attention on one element, say E_2 . It will be shown in §10 that a fluid element with an excess of vorticity is accelerated in the direction of the positive y -axis with an acceleration $\Gamma^{-1} \iint \{v'(x, y)\}^2 \zeta'_0' dx dy$, where $\zeta'_0(y)$ is the gradient of vorticity of the main flow, $v'(x, y)$ is the component of the disturbing velocity perpendicular to the direction of flow, and Γ is the total strength of the vortex filaments corresponding to the disturbance. Examining the signs of the various quantities in the acceleration formula, we can easily see that E_2 is accelerated toward a region of higher vorticity if the gradient of vorticity does not change sign anywhere in the fluid. Thus, E_2 is accelerated toward L_2 . A similar consideration holds for the element E_1 . Hence, in either case, *the fluid element is returned to the layer where it belonged* (by the acceleration due to its interaction with other vortex filaments). *The motion is therefore stable when the gradient of the vorticity does not vanish.*

When there is an extremum of vorticity, an interchange of fluid elements on opposite sides of the extremum does not give rise to an excess or a defect of vorticity. Furthermore, the gradient of vorticity vanishes there, and has opposite signs on opposite sides of that layer. It can easily be seen from the above acceleration formula that the restoring tendency mentioned above is largely impaired in such a case. Thus, exchanged fluid elements are not as strongly forced back by the action discussed above. Such an exchange constitutes a disturbance because there is an exchange of momentum. Thus, a disturbance may tend to persist and perhaps to augment. The motion is not necessarily stable.

⁷ Cf. discussions of the vorticity transfer theory of turbulence in his general lecture at the Fourth International Congress for Applied Mechanics [19]. Some developments in that direction were continued by C. B. Millikan (unpublished).

The above discussion is based on very general considerations and does not depend on the consideration of a periodic wavy disturbance as used in the mathematical analysis. We shall now support the above argument by considering a neutral wavy disturbance, with the understanding that if such a disturbance can persist (except for the exceptional case of infinite wave-length and zero phase velocity), the motion is presumably unstable. From these considerations, the importance of viscosity in the inner friction layer will also be brought out.

Let us consider an observer moving with the phase velocity of a neutral wavy disturbance. He will observe a stationary pattern of the flow (see Fig. 7).⁸ Closed stream lines are inevitable unless the disturbance has no v -component of velocity in the critical layer $w=c$, for the flows on opposite sides of the critical layer are in opposite directions relative to the observer. It appears unlikely that the v -component of the disturbance should be zero throughout that layer. Indeed, it has been shown to be impossible mathematically.⁹ Thus, whenever a neutral disturbance persists, it involves a steady exchange of fluid elements on opposite sides of the critical layer.

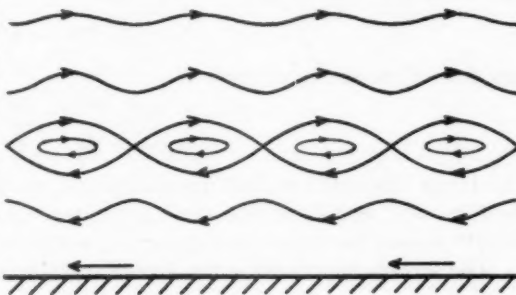


FIG. 7. Stream lines of a neutral disturbance as observed by an observer moving with the wave velocity.

If the effect of viscosity is to be negligible, fluid elements on the same stream line must have the same vorticity. If the gradient of vorticity of the main flow is zero or small near the critical point, it is easy to compensate this small difference of vorticity by the vorticity of the superposed flow, while the "scale" of disturbance [as measured in order of magnitude by $u' / (\partial u' / \partial y)$] remain the same as that of the main flow. It is thus *not impossible* to find a neutral disturbance for which the effect of viscosity is negligible. The motion may be unstable.

On the other hand, if the gradient of vorticity of the main flow is finite, the superposed small disturbance must also give a finite gradient of vorticity. This means that the "scale" of the disturbance must be very small in the critical layer. The diffusion of vorticity by the effect of viscosity is then inevitable. It is thus *impossible* to find a neutral disturbance for which the effect of viscosity is negligible. The motion is inertially stable.

10. Acceleration of vortices in a non-uniform field of vorticity. In the foregoing physical interpretation of inertial instability, we have considered the acceleration of an element of fluid in a two-dimensional parallel flow when this element of fluid does not have the same vorticity as the surrounding layer. We are now going to derive the explicit formula for the acceleration. The derivation shall be made in two different ways: (1) by kinematical considerations (using vorticity theorems); (2) by considera-

⁸ This figure is due to Lord Kelvin (loc. cit.). He pointed out that the facts discussed here are "surprising," but did not attempt to explain their connection with the mechanism of hydrodynamic stability.

⁹ This follows at once from Rayleigh's original results, if we apply it to the region between this layer and the solid wall (cf. Tollmien, loc. cit., 1935).

tions of the pressure gradient. In either method, we shall consider a perfect fluid in accordance with the stability problem under consideration.

1) *First derivation, by kinematical considerations.* For definiteness, let us consider a two-dimensional flow between two solid walls, which we shall take to be $y = \pm b$. Let the velocity components of the main flow be

$$U = w(y), \quad V = 0, \quad (10.1)$$

and those of the secondary flow be

$$u' = u'(x, y), \quad v' = v'(x, y). \quad (10.2)$$

The distribution of vorticity of the main flow is

$$\zeta_0 = \zeta_0(y) = -w'(y), \quad (10.3)$$

and that of the secondary flow is

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}. \quad (10.4)$$

The latter distribution shall approximate a vortex "at" the point (ξ_0, η_0) . Thus, if the signs of ζ' and ζ_0 are the same (or opposite), we have essentially a small element of fluid having an excess (or a defect) of vorticity near the point (ξ_0, η_0) .

The stream function for the secondary flow is

$$\psi'(x, y) = -\frac{1}{2\pi} \iint \zeta'(\xi, \eta) G(x, y; \xi, \eta) d\xi d\eta, \quad (10.5)$$

with

$$\left. \begin{aligned} u'(x, y) &= -\frac{\partial \psi'}{\partial y} = \frac{1}{2\pi} \iint \zeta'(\xi, \eta) \frac{\partial}{\partial y} G(x, y; \xi, \eta) d\xi d\eta, \\ v'(x, y) &= \frac{\partial \psi'}{\partial x} = -\frac{1}{2\pi} \iint \zeta'(\xi, \eta) \frac{\partial}{\partial x} G(x, y; \xi, \eta) d\xi d\eta. \end{aligned} \right\} \quad (10.6)$$

In these expressions, the integrals are extended over the whole region between the planes. The function $G(x, y; \xi, \eta)$ is the Green's function of the first kind for the region under consideration. It is defined by the following conditions:

$$\left. \begin{aligned} \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} &= 0 \text{ except at } (\xi, \eta), \\ G(x, y; \xi, \eta) &\sim -\log \{(x - \xi)^2 + (y - \eta)^2\}^{1/2} \text{ near } (\xi, \eta), \\ G(x, y; \xi, \eta) &= 0 \text{ over the solid boundaries.} \end{aligned} \right\} \quad (10.7)$$

As is well-known, it has the reciprocity property

$$G(x, y; \xi, \eta) = G(\xi, \eta; x, y). \quad (10.8)$$

For the case of a channel, it is given by the real part of

$$\begin{aligned} f(z) &= -\left\{ \log \operatorname{sh} \frac{\pi}{4b} (z - z_0) - \log \operatorname{ch} \frac{\pi}{4b} (z - \bar{z}_0) \right\}, \\ (z &= x + iy, z_0 = \xi + i\eta, \bar{z}_0 = \xi - i\eta). \end{aligned} \quad (10.9)$$

Let us now consider the behavior of a particular element of fluid at (ξ, η) having an excess (or a defect) of vorticity corresponding to the secondary flow (10.5) and (10.6). It causes a distortion of the main vorticity distribution as indicated in Fig. 8. After a very small interval of time δt the vorticity at the point (x, y) is changed by the amount

$$\delta \zeta(x, y) = -v'(x, y)\delta t \zeta'_0(y), \quad (10.10)$$

because it is replaced by a fluid element from below, which retains its original vorticity. This change produces an effect at the "vortex," i.e., at the element of fluid under

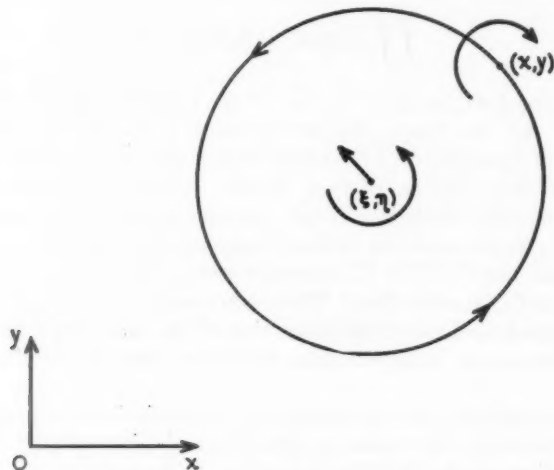


FIG. 8. Acceleration of vortices in a non-uniform field of vorticity ($\zeta'_0(y) > 0$, $\Gamma > 0$).

consideration at (ξ, η) . It can be easily seen that the effect is a small velocity with components

$$\left. \begin{aligned} \delta u(\xi, \eta) &= \frac{1}{2\pi} \iint \frac{\partial}{\partial \eta} G(x, y; \xi, \eta) \delta \zeta(x, y) dx dy, \\ \delta v(\xi, \eta) &= -\frac{1}{2\pi} \iint \frac{\partial}{\partial \xi} G(x, y; \xi, \eta) \delta \zeta(x, y) dx dy, \end{aligned} \right\} \quad (10.11)$$

the integrals being extended over the whole region between the planes. Dividing these quantities by δt and passing to the limit $\delta t \rightarrow 0$, we have the following components of acceleration at the point (ξ, η) :

$$\left. \begin{aligned} a_x(\xi, \eta) &= -\frac{1}{2\pi} \iint \frac{\partial}{\partial \eta} G(x, y; \xi, \eta) v'(x, y) \zeta'_0(y) dx dy, \\ a_y(\xi, \eta) &= \frac{1}{2\pi} \iint \frac{\partial}{\partial \xi} G(x, y; \xi, \eta) v'(x, y) \zeta'_0(y) dx dy. \end{aligned} \right\} \quad (10.12)$$

Let us first consider the y -component of this acceleration. From the special form in which x and ξ enter into the Green's function [cf. (10.9)], we can also write

$$a_y(\xi, \eta) = -\frac{1}{2\pi} \iint v'(x, y) \zeta'_0(y) \frac{\partial}{\partial x} G(x, y; \xi, \eta) dx dy. \quad (10.13)$$

If we multiply this equation by $\zeta'(\xi, \eta)$ and integrate over the whole region, we have the final formula

$$\iint a_y(\xi, \eta) \zeta'(\xi, \eta) d\xi d\eta = \iint \{v'(x, y)\} \zeta'_0(y) dx dy, \quad (10.14)$$

upon using (10.6). Before discussing its significance, let us first notice that

$$\iint a_x(\xi, \eta) \zeta'(\xi, \eta) d\xi d\eta = 0 \quad (10.15)$$

if $\zeta'(\xi, \eta)$ is an even function of $\xi - \xi_0$, i.e., if the vorticity distribution $\zeta'(\xi, \eta)$ has a symmetry about the line $\xi = \xi_0$. For then $v'(x, y)$ is an odd function of $x - \xi_0$ and $a_x(\xi, \eta)$ is an odd function of $\xi - \xi_0$, both being the consequence of the fact that $G(x, y; \xi, \eta)$ is an even function of $x - \xi$. Hence, we have the conclusion.

If we recall that the vorticity $\zeta'(\xi, \eta)$ is spread over a small region, we may take $\Gamma = \iint \zeta'(\xi, \eta) d\xi d\eta$ as the strength of the "superposed vortex." If we divide the left-hand side of (9.14) and (9.15) by Γ , we may consider the results as giving the components of the "average acceleration." The x -component of acceleration vanishes; the sign of the y -component depends upon the sign of the superposed vortex and the sign of $\zeta'_0(y)$. This component of acceleration is the one used in the above physical considerations.

It should be mentioned that in considering the stability of a motion we deal with a vortex pair. Although this makes it difficult to obtain a compact formula for the average accelerations of the individual vortices, a kinematical consideration such as that given above (cf. Fig. 8) shows that the general tendency is not changed. Furthermore, the two vortices are soon separated, because they are situated in layers of different mean velocity.

Another point should be mentioned. If we notice the tendency for the main vorticity to be swung around the secondary vortex, there is an acceleration of every element of fluid toward the vortex. Whatever this acceleration may be, it is expected to be of minor importance, because the effect is spread out over the whole field. This point will be brought out clearly in the following derivation of (10.14), where we shall study the whole phenomenon from the point of view of pressure forces. The acceleration will be identified with the negative of the pressure gradient divided by the density of the fluid, because the effect of viscosity has been neglected. Thus, if we can calculate the pressure disturbance corresponding to a given velocity disturbance, the left-hand side of (10.14) can be calculated.

2) *Second derivation, by consideration of pressure forces correlated with vorticity fluctuations.* To calculate the pressure distribution from a given velocity distribution, we use for the pressure a differential equation of Poisson's type obtained by taking the divergence of the equations of motion. Thus, if the equations of motion are¹⁰

¹⁰ The usual notation is used: x_i ($i = 1, 2, 3$) are the coordinates, u_i are the components of velocity, p is the pressure, and ρ is the density of the fluid. Summation over a repeated index is understood. For a discussion of this type, see Lichtenstein's book [26].

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i, \quad (i = 1, 2, 3), \quad (10.16)$$

and the equation of continuity is

$$\partial u_i / \partial x_i = 0, \quad (10.17)$$

we have

$$\Delta(p/\rho) = -\sigma, \quad (10.18)$$

where

$$\sigma = \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} = e_{ij} e_{ij} - \omega_{ij} \omega_{ij} = 2 \left\{ \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} + \frac{\partial(u_2, u_3)}{\partial(x_2, x_3)} + \frac{\partial(u_3, u_1)}{\partial(x_3, x_1)} \right\}, \quad (10.19)$$

and

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \quad (10.20)$$

are the components of deformation and of rotation. If we integrate (10.18) under proper boundary conditions, σ being known at the *initial* instant, we obtain the *initial* distribution of pressure. The initial acceleration field is then obtained from (10.16) as the negative gradient of the pressure, if we neglect the effect of viscosity.

For a perfect fluid, the only boundary condition at a solid wall is

$$u_i n_i = 0, \quad (10.21)$$

where n_i is the outward normal of the boundary surface. If we multiply (10.16) by n_i , neglecting the effect of viscosity, we have

$$- \frac{1}{\rho} \frac{\partial p}{\partial n} = V_0 \frac{\partial u_i}{\partial s} n_i, \quad (10.22)$$

where V_0 is the velocity along a stream line on the boundary, and ds is an element of its arc. If we write

$$u_i = V_0 l_i, \quad \frac{\partial u_i}{\partial s} = V_0 \frac{\partial l_i}{\partial s} + l_i \frac{\partial V_0}{\partial s},$$

where l_i are the direction cosines of the velocity over the boundary surface, we have

$$- \frac{1}{\rho} \frac{\partial p}{\partial n} = \frac{V_0^2}{R}, \quad (10.23)$$

where R is the radius of curvature of the stream line, $R^{-1} = n_i \partial l_i / \partial s$. This relation expresses the balance of pressure and centrifugal force. With a given distribution of velocity, the right-hand side is known. We have thus a potential problem of the second kind for the pressure.

Two-dimensional flow between parallel solid walls. Returning to the problem at hand, we have the very simple boundary condition

$$\partial p / \partial y = 0 \quad \text{at} \quad y = \pm b. \quad (10.24)$$

Since the main motion is a two-dimensional parallel motion, we have

$$u_1 = w(y) + u'(x, y), \quad u_2 = v'(x, y), \quad u_3 = 0, \quad (10.25)$$

where $w(y)$ represents the main flow, and u' and v' give a secondary flow approximating a vortex. Equation (10.18) becomes

$$\frac{1}{\rho} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) = -\sigma, \quad \sigma = 2 \left\{ w'(y) \frac{\partial v'}{\partial x} - \frac{\partial(u', v')}{\partial(x, y)} \right\}. \quad (10.26)$$

We note that we can write $\sigma = \sigma_1 + \sigma_2$, where

$$\sigma_1 = -2 \frac{\partial(u', v')}{\partial(x, y)}, \quad \sigma_2 = -2 \zeta_0 \frac{\partial v'}{\partial x}. \quad (10.27)$$

σ_1 depends upon the structure of the secondary vortex itself, and σ_2 depends upon its interaction with the main flow. We shall also separate the pressure into two parts and require them to satisfy (10.24) separately. Thus,

$$\left. \begin{aligned} p &= p_1 + p_2, & \sigma &= \sigma_1 + \sigma_2, \\ \frac{1}{\rho} \left(\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} \right) &= -\sigma_1, & \frac{\partial p_1}{\partial y} &= 0 \text{ at } y = \pm b, \\ \frac{1}{\rho} \left(\frac{\partial^2 p_2}{\partial x^2} + \frac{\partial^2 p_2}{\partial y^2} \right) &= -\sigma_2, & \frac{\partial p_2}{\partial y} &= 0 \text{ at } y = \pm b. \end{aligned} \right\} \quad (10.28)$$

We can reduce our problem to that of the *first* kind by looking for the acceleration $a_y(x, y)$ in the y -direction, $a_y = -(1/\rho) \partial p / \partial y$. If we differentiate (10.28) with respect to y , we have

$$\left. \begin{aligned} a_y &= \alpha_1 + \alpha_2, \\ \frac{\partial^2 \alpha_1}{\partial x^2} + \frac{\partial^2 \alpha_1}{\partial y^2} &= \frac{\partial \sigma_1}{\partial y}, & \frac{\partial^2 \alpha_2}{\partial x^2} + \frac{\partial^2 \alpha_2}{\partial y^2} &= \frac{\partial \sigma_2}{\partial y}, \end{aligned} \right\} \quad (10.29)$$

with $\alpha_1 = 0, \alpha_2 = 0$ at $y = \pm b$. The x -component of acceleration is zero, from symmetry considerations. The solutions of (10.29) are

$$\left. \begin{aligned} \alpha_1(x, y) &= -\frac{1}{2\pi} \iint G(x, y; \xi, \eta) \left(\frac{\partial \sigma_1}{\partial y} \right)_{\xi, \eta} d\xi d\eta, \\ \alpha_2(x, y) &= \frac{1}{2\pi} \iint G(x, y; \xi, \eta) \left(\frac{\partial \sigma_2}{\partial y} \right)_{\xi, \eta} d\xi d\eta, \end{aligned} \right\} \quad (10.30)$$

where the integrals are extended over the whole region between the planes. These formulae give the distribution of acceleration. Actually, it is more convenient to deal with the integrated quantities

$$I_1 = \iint \alpha_1(x, y) \zeta_0(x, y) dx dy, \quad (10.31)$$

$$I_2 = \iint \alpha_2(x, y) \zeta_0(x, y) dx dy, \quad (10.32)$$

$$J_1 = \iint \alpha_1(x, y) \zeta'(x, y) dx dy, \quad (10.33)$$

$$J_2 = \iint \alpha_2(x, y) \zeta'(x, y) dx dy. \quad (10.34)$$

The first two integrals correspond to the accelerations of the main flow by the secondary flow itself and by the interaction; the latter two quantities correspond to the accelerations of the secondary flow by itself and by the interaction. It can be verified (as will be done presently) that

$$I_1 = - \iint v'^2 \frac{d\zeta_0}{dy} dx dy, \quad (10.35) \quad I_2 = 0, \quad (10.36)$$

$$J_1 = 0, \quad (10.37) \quad J_2 = \iint v'^2 \frac{d\zeta_0}{dy} dx dy. \quad (10.38)$$

We note that (10.38) is essentially a reproduction of the formula (10.14), the significance of which has been discussed above. The integral I_1 is equal to the negative of J_2 . This is the above-mentioned acceleration distributed among the fluid elements throughout the field. It is therefore relatively unimportant. Thus, all the statements made in the last section have been verified, if we can verify (10.35)–(10.38).

Verification of (10.35)–(10.38). To verify these equations, let us first examine the behavior of the quantities u' , v' , $\partial p/\partial x$, $\partial p/\partial y$ for large values of x . From the expression (10.9) for the Green's function, we see that if ζ' vanishes sufficiently rapidly as x becomes infinite, we have

$$u' = O(|x|^{-3}), \quad v' = O(|x|^{-2}), \quad (10.39)$$

for large values of x . From the equations of motion, we then find that

$$\frac{\partial p}{\partial x} = O(u') = O(|x|^{-3}), \quad \frac{\partial p}{\partial y} = O(v') = O(|x|^{-2}). \quad (10.40)$$

This will assure the convergence of the integrals involved and the validity of the steps taken in the following transformations.

In the first method of derivation, we have been mainly concerned with J_2 . We shall therefore consider it first. Referring to (10.30) and (10.5), we see that

$$J_2 = \iint \psi'(\xi, \eta) \left(\frac{\partial \sigma_2}{\partial y} \right)_{\xi, \eta} d\xi d\eta.$$

If we now introduce the value of σ_2 as given by (10.27) and replace (ξ, η) by (x, y) , we have

$$J_2 = -2 \iint \psi'(x, y) \frac{\partial^2}{\partial x \partial y} (v' \zeta_0) dx dy.$$

On integrating by parts with respect to x , we obtain

$$J_2 = 2 \iint v' \frac{\partial}{\partial y} (v' \zeta_0) dx dy = \iint \left\{ \frac{\partial}{\partial y} (v'^2 \zeta_0) + v'^2 \frac{d\zeta_0}{dy} \right\} dx dy.$$

The result (10.38) or (10.14) is thereby verified.

Similar calculations can be carried out for the integrals I_1 , I_2 , and J_1 . Thus,

$$\begin{aligned} I_1 &= \iint \alpha_1 \zeta_0' dx dy = \iint w(y) \frac{\partial \alpha_1}{\partial y} dx dy \\ &= \iint w \left(\sigma_1 + \frac{1}{\rho} \frac{\partial^2 p_1}{\partial x^2} \right) dx dy, \end{aligned}$$

by (10.28). If we note that

$$w\sigma_1 = -2w \frac{\partial(u', v')}{\partial(x, y)} = 2 \frac{\partial}{\partial x} \left(wv' \frac{\partial u'}{\partial y} \right) - 2 \frac{\partial}{\partial y} \left(wv' \frac{\partial u'}{\partial x} \right) - \frac{dw}{dy} \frac{\partial^2 v'}{\partial y^2},$$

the above integral is easily transformed into the form (10.35). Following an exactly analogous process, we have

$$I_2 = \iint w \left(\sigma_2 + \frac{1}{\rho} \frac{\partial^2 p_2}{\partial x^2} \right) dx dy = 0,$$

when we make use of (10.27). The integral J_1 has also the significance that it is the effect of the solid boundaries upon a general flow $\psi'(x, y)$ consistent with (10.40), because it is independent of $w(y)$. Using (10.30), (10.5), and (10.27), we have

$$\begin{aligned} J_1 &= \iint \alpha_1 \zeta_1' dx dy = \iint \psi'(\xi, \eta) \left(\frac{\partial \sigma_1}{\partial y} \right)_{\xi, \eta} d\xi d\eta \\ &= -2 \iint u' \frac{\partial(u', v')}{\partial(x, y)} dx dy. \end{aligned}$$

If we note that

$$u' \frac{\partial(u', v')}{\partial(x, y)} = \frac{\partial}{\partial y} \left(u'v' \frac{\partial u'}{\partial x} \right) - \frac{\partial}{\partial x} \left(u'v' \frac{\partial u'}{\partial y} \right),$$

we see that $J_1=0$. The results (10.35)–(10.38) are thereby verified. We have thus completed the investigations indicated at the beginning of this section.

NOTE ADDED IN PROOF. In a very early work, [Phil. Trans. Roy. Soc. London (A) 215, 23–26 (1915)] G. I. Taylor gave a physical interpretation of Rayleigh's results on the stability of the laminar motion of an inviscid fluid, based on momentum considerations. He also indicated clearly that a motion, stable according to Rayleigh's criterion, may be unstable through the effect of viscosity.

(To be continued)

ON THE VIBRATIONS OF THE ROTATING RING*

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1. Introduction. An interesting addition to the group of problems dealing with thin elastic rings is the analysis of the vibration of a circular ring which is rotating with constant speed about its geometric axis. In this paper, the small bending vibrations of the unconstrained ring are analyzed and the frequencies at which such vibrations can occur are determined. For various problems of the partially constrained ring, it is shown that the "free vibrations" differ essentially in character from those of the free ring, exhibiting a group of natural modes characterized by linear combinations of trigonometric functions. The forced vibrations of both the free and supported rings are also treated.

2. The dynamic equations. The three equations needed to specify completely the plane motion of an element of a ring, such as the one shown in Fig. 1, are derived from a consideration of the forces and moments acting on the element and the components of acceleration of the element. The summation of forces along $o'a'$, the summation of moments about o' , and the summation of moments about a' , lead to this required set of equations, which is

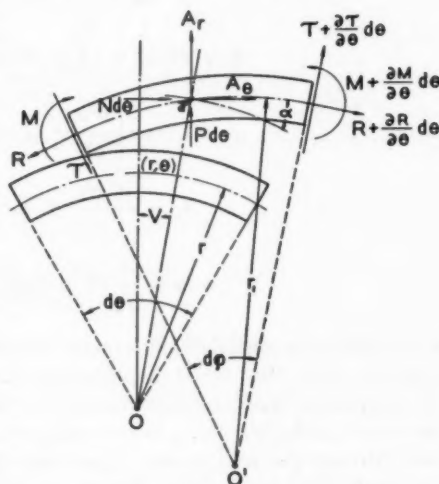


FIG. 1. Element of ring in initial and distorted positions. Center of rotation is o .

$$-Rd\phi + \left[\frac{\partial \tau}{\partial \theta} + \left(P - \frac{k}{r} A_r \right) \cos(\alpha - v) - \left(N - \frac{k}{r} A_\theta \right) \sin(\alpha - v) \right] d\theta = 0, \quad (1)$$

$$\frac{\partial R}{\partial \theta} - \frac{1}{r_1} \frac{\partial M}{\partial \theta} + \left(N - \frac{k}{r} A_\theta \right) \cos(\alpha - v) + \left(P - \frac{k}{r} A_r \right) \sin(\alpha - v) = 0, \quad (2)$$

$$r\tau = \frac{\partial M}{\partial \theta} - rk \frac{a}{c} \frac{\partial^2(\alpha - v)}{\partial t^2} = 0. \quad (3)$$

Here, the notations are as follows: θ is a polar coordinate of a point in the undeformed ring, referred to axes rotating with the ring; r and r_1 are respectively the radii of curvature of the undistorted and distorted rings; b , h and I are respectively the width, thickness and cross-sectional moment of inertia of the ring; R , τ and M are respec-

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tively the tensile force, shearing force and bending moment, as shown; N/r and P/r are the components of the external forces in the directions of the tangent and normal in the undistorted state, as shown; A_r and A_θ are the components of acceleration in the directions of the tangent and normal in the undistorted state, as shown; E is the elastic modulus; ρ is the density; $k = \rho b h r^2$; $c = E b h$; $a = E I / r^2$.

In Eqs. (1)–(3) the corrections, arising from the Poisson strains, for the moment of inertia expressions, etc., have been omitted as usual. The formulas needed to supplement the above equations are

$$M = \frac{EI}{r} \left(u_0 + u + \frac{\partial^2 u}{\partial \theta^2} \right), \quad (4a) \quad \frac{1}{r_1} = \frac{1}{r} \left(1 - u_0 - u - \frac{\partial^2 u}{\partial \theta^2} \right), \quad (4b)$$

$$R = E b h e_\theta = c \left(u_0 + u + \frac{\partial v}{\partial \theta} \right), \quad (4c) \quad \alpha = \frac{\partial u}{\partial \theta}, \quad (4d)$$

$$A_r = r \left[\frac{\partial^2 u}{\partial t^2} - \omega^2 (1 + u_0 + u) - 2\omega \frac{\partial v}{\partial t} \right], \quad (4e)$$

$$A_\theta = r \left[\frac{\partial^2 v}{\partial t^2} + 2\omega \frac{\partial u}{\partial t} - \omega^2 v \right], \quad (4f)$$

$$d\phi = \frac{r}{r_1} (1 + e_\theta) d\theta = \left(1 + \frac{\partial v}{\partial \theta} - \frac{\partial^2 u}{\partial \theta^2} \right) d\theta, \quad (4g)$$

where ru_0 is the radial displacement from the rest position to the rotating equilibrium position, ru is the radial displacement from the rotating equilibrium position, rv is the tangential displacement relative to the rotating axes,¹ e_θ is the tangential strain ($e_\theta = u_0 + u + \partial v / \partial \theta$), and ω is the constant angular velocity of the ring. Equations (4a) and (4b) are the well known expressions for the bending moment and curvature, respectively, of a bent ring;² (4c) is a one-dimensional form of Hooke's law; (4d) is the rotational displacement of the element, and is found by inspection of Fig. 1; (4e) and (4f) are the expressions for the radial and tangential components of acceleration, when u and v are referred to a rotating coordinate system;³ and g is obtained from Fig. 1 and Eq. (4b).

The value of u_0 is obtained by writing $u = v = 0$ in Eq. (1). After substitutions from Eqs. (4), it becomes $E b h u_0 = k \omega^2 (1 + u_0)$, or

$$u_0 = (1 + u_0) k \omega^2 / c. \quad (5)$$

Throughout this analysis, we shall consider only those vibrations for which u and v are small compared to unity. We are therefore justified in disregarding terms in u^2 , uv , etc., as compared to u or v . In the limit, that is, as the amplitude of u and v tend to zero, the equations obtained in this manner would be exact. However, the equations so obtained would still be encumbered by terms of the type $u_0^2 u$, $u_0^2 v$, . . . , in addition to those found below in Eqs. (6) and (7). We also neglect these terms since

¹ This rather unconventional notation is used to provide u_0 , u and v with dimensionless properties and thus produce somewhat less cumbersome equations.

² S. P. Timoshenko, *Strength of materials*, D. Van Nostrand Co., New York, 1930, p. 459.

³ These are easily deduced from the vector forms given in L. Page, *Introduction to theoretical physics*, D. Van Nostrand Co., New York, 1941, p. 103.

they can produce no qualitative changes in the results and since they will drop out a yway when the procedure leading to Eq. (9a) is introduced. Finally, since the amplitudes of P and N must obviously vanish when u and v tend to zero, terms in Pu , Pv , \dots , must also be omitted.

Following this procedure, using Eq. (3) to eliminate τ , and substituting from Eqs. (4) when necessary, we obtain from Eqs. (1) and (2)

$$P - c \left(u + \frac{\partial v}{\partial \theta} \right) - a \left(\frac{\partial^4 u}{\partial \theta^4} + \frac{\partial^2 u}{\partial \theta^2} \right) + k \omega^2 \left(\frac{\partial^2 u}{\partial \theta^2} + u - \frac{\partial v}{\partial \theta^2} \right) \\ = k \left[\frac{\partial^2 u}{\partial t^2} - 2\omega \frac{\partial v}{\partial t} - \frac{a}{c} \frac{\partial^4 u}{\partial \theta^2 \partial t^2} + \frac{a}{c} \frac{\partial^2 v}{\partial \theta \partial t^2} \right], \quad (6)$$

$$N + c \frac{\partial}{\partial \theta} \left(u + \frac{\partial v}{\partial \theta} \right) - a \left(\frac{\partial^3 u}{\partial \theta^3} + \frac{\partial u}{\partial \theta} \right) + k \omega^2 \frac{\partial u}{\partial \theta} = k \left[\frac{\partial^2 v}{\partial t^2} + 2\omega \frac{\partial u}{\partial t} \right]. \quad (7)$$

We may easily arrive at a single equation in u only by performing on Eq. (6) the operation L where,

$$L = c \frac{\partial^2}{\partial \theta^2} + k \frac{\partial^2}{\partial t^2}, \quad (8)$$

solving Eq. (7) for $L(v)$, and substituting the expression found by the latter step into that found by the former.⁴ We utilize the abbreviations,

$$s = t\sqrt{a/k}, \quad \mu = \omega\sqrt{k/a}, \quad \epsilon = a/c = h^2/12r^2,$$

and the equation resulting from the foregoing procedure takes the form,

$$\left\{ \left[\left(\frac{\partial^2}{\partial \theta^2} - 1 \right) \frac{\partial^2}{\partial s^2} + 4\mu \frac{\partial^2}{\partial \theta \partial s} + \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right) - \mu^2 \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 3 \right) \right] \right. \\ \left. - \epsilon \left[\frac{\partial^4}{\partial s^4} + \left\{ 2 \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right) - \mu^2 \left(\frac{\partial^2}{\partial \theta^2} - 3 \right) \right\} \frac{\partial^2}{\partial s^2} \right. \right. \right. \\ \left. \left. + 2\mu \left(\frac{\partial^2}{\partial \theta^2} + 1 - 2\mu^2 \right) \frac{\partial^2}{\partial \theta \partial s} + (\mu^4 - \mu^2) \frac{\partial^2}{\partial \theta^2} - \mu^2 \frac{\partial^4}{\partial \theta^4} \right] \right. \\ \left. + \epsilon^2 \left[\frac{\partial^6}{\partial \theta^2 \partial s^4} + 2\mu \frac{\partial^4}{\partial \theta \partial s^3} - \left(\mu^2 \frac{\partial^2}{\partial \theta^2} - \frac{\partial^4}{\partial \theta^4} - \frac{\partial^2}{\partial \theta^2} \right) \frac{\partial^2}{\partial s^2} \right] \right\} u \\ = \frac{\partial^2}{\partial \theta^2} \left(\frac{P}{a} \right) + \frac{\partial}{\partial \theta} \left(\frac{N}{a} \right) - \epsilon \left[\frac{\partial^2}{\partial s^2} \left(\frac{P}{a} \right) + 2\mu \frac{\partial}{\partial s} \left(\frac{N}{a} \right) \right] + \epsilon^2 \frac{\partial^2}{\partial \theta \partial s^2} \left(\frac{N}{a} \right). \quad (9)$$

This equation governs the motion of the ring, provided the driving functions N and P do not imply that the deformations be large. The validity of this equation may be partially checked by considering a physically trivial problem. We consider the freely spinning ring (no supports) and suppose N to P to vanish. Under these conditions, the motion of the ring which is initially not deformed from its equilibrium shape is given by $u = (\alpha + \beta s) \cos(\theta + \mu s)$, $v = -(\alpha + \beta s) \sin(\theta + \mu s)$. That is, the ring moves as

⁴ A similar procedure will provide an analogous equation in v .

a rigid body with (dimensionless) angular velocity μ and translational velocity β . Eq. (9) must and does allow this solution for all α and β .

In each of those problems to be considered, Eq. (9), in its present form, leads to a solution which is simple in form but which requires the solution of unnecessarily long algebraic equations. This computational work may be eliminated at the expense of small errors in accuracy when we consider only rings for which h/r is small. In this case, terms in Eq. (9) of order ϵ and ϵ^2 may be neglected compared to those of order one. This leads to the following equation, which is exact for the limiting case where u, v, ϵ tend to zero:

$$\begin{aligned} L_1(u) &= \left[\mu^2 \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 3 \right) - \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right)^2 - 4\mu \frac{\partial^2}{\partial \theta \partial s} - \left(\frac{\partial^2}{\partial \theta^2} - 1 \right) \frac{\partial^2}{\partial s^2} \right] u \\ &= - \frac{\partial^2}{\partial \theta^2} \left(\frac{P}{a} \right) - \frac{\partial}{\partial \theta} \left(\frac{N}{a} \right). \end{aligned} \quad (9a)$$

The analogous equation in v is

$$L_1(v) = \frac{N}{a} + \frac{\partial}{\partial \theta} \left(\frac{P}{a} \right). \quad (9b)$$

In each of the problems to follow, the expressions obtained from Eqs. (9a) and (9b) for the natural frequencies and amplitudes are valid to within errors of order h^2/r^2 for those frequencies of order $(a/k)^{1/2}$. These vibrations may be termed "bending vibrations" since they are essentially inextensional forms of motion. When the frequencies are of order $(c/k)^{1/2}$, accurate results may be obtained by direct use of Eq. (9). In this paper, we shall use only Eq. (9a) or (9b) since all of the characteristics of the effects of rotation on the behavior of the ring will appear in the solutions so obtained. The one exception to this statement is found in connection with the dilatory vibrations. This type of motion can not be predicted by Eq. (9a) or (9b) because these equations are those for essentially inextensional motions.⁵ We shall, then, when investigating this mode, refer to Eq. (9). For this mode, u and v are independent of θ as must be those parts of P and N which excite such a motion. Hence Eq. (9) reduces for this case to

$$L_0(u) = \left[\frac{\partial^4}{\partial s^4} + \left(3\mu^2 + \frac{c}{a} \right) \frac{\partial^2}{\partial s^2} \right] u = \frac{\partial^2}{\partial s^2} \left(\frac{P}{a} \right) + 2\mu \frac{\partial}{\partial s} \left(\frac{N}{a} \right), \quad (9c)$$

or, in terms of v ,

$$L_0(v) = \left(\frac{\partial^2}{\partial s^2} + \frac{c}{a} - \mu^2 \right) \frac{N}{a} - 2\mu \frac{\partial}{\partial s} \frac{P}{a}. \quad (9d)$$

3. The unconstrained ring. Investigation of the solutions of Eqs. (9a) or (9b) of the form

$$u_n = U_n \cos n(\theta - \beta_n s) + W_n \sin n(\theta - \beta_n s) \quad (10)$$

yields, when N and P vanish identically,

⁵ Equation (9a) may also be derived from Eqs. (6) and (7) by use of the assumption that $u + \partial v / \partial \theta \ll u$; this excludes the dilatory motion.

$$\beta_n = -\frac{2\mu}{n^2 + 1} \pm \frac{(n^2 - 1)}{n^2 + 1} \sqrt{\mu^2 + n^2 + 1} = -q_n \pm p_n, \quad (10a)$$

where p_n and q_n are self-defining.

If we further apply a set of initial conditions, such as

$$u_n = U_n \cos n\theta + W_n \sin n\theta \quad \text{and} \quad \frac{\partial u_n}{\partial s} = 0, \quad \text{at } s = 0,$$

to the two solutions defined for a given n by Eqs. (10) and (10a), namely

$$u_n = U_n [a_n \cos n(\theta + q_n s - p_n s) + b_n \cos n(\theta + q_n s + p_n s)] \\ + W_n [c_n \sin n(\theta + q_n s - p_n s) + d_n \sin n(\theta + q_n s + p_n s)], \quad (11)$$

we obtain

$$u_n = U_n \left[\cos n(\theta + q_n s) \cos n p_n s + \frac{q_n}{p_n} \sin n(\theta + q_n s) \sin n p_n s \right] \\ + W_n \left[\sin n(\theta + q_n s) \cos n p_n s - \frac{q_n}{p_n} \cos n(\theta + q_n s) \sin n p_n s \right]. \quad (12)$$

Each term in the foregoing bracket defines a possible free vibration of the ring which is unconstrained at all points against either radial or tangential displacement. Each of these terms may be interpreted as defining a "normal mode" of vibration, wherein a sinusoidal deformation of angular frequency $n p_n$ travels with respect to the rotating axes at an angular velocity* $-q_n$. The "nodal points" thus move with respect to coordinates fixed in the ring. The terms "normal mode" and "nodal point" have been used somewhat loosely here, but they adhere to the usual definitions of the terms if the motion is described relative to axes rotating with velocity $\Omega_n = \omega(n^2 - 1)/(n^2 + 1)$.

For the stationary ring, the value of the angular frequency reduces to

$$(a/k)^{1/2} n p_n = n(n^2 - 1) [a/k(n^2 + 1)]^{1/2},$$

which is in agreement with previously derived results.⁷

A solution to Eq. (9a) may be obtained for arbitrary initial distributions of radial deflection U_0 and radial velocity U'_0 , provided these initial conditions do not imply an extensional motion. The restriction

$$\int_0^{2\pi} U_0(\theta) d\theta = \int_0^{2\pi} U'_0(\theta) d\theta = 0 \quad (13)$$

is certainly sufficient to insure this provision since it, together with the continuity requirements,

$$\int_0^{2\pi} \frac{\partial v}{\partial \theta} d\theta = \int_0^{2\pi} \frac{\partial^2 v}{\partial \theta \partial s} = 0$$

allows $u + \partial v / \partial \theta$ to vanish for all θ , at and shortly after time $t = 0$.

The restriction defined by Eq. (13) together with the requirement that u be continuous, implies that U_0 and U'_0 may be expanded in Fourier series in which the

* p_n and q_n are, of course, dimensionless quantities which define the angular velocities.

⁷ J. P. Den Hartog, *Mechanical vibrations*, McGraw-Hill, New York, 1940, p. 123.

constant term vanishes and in which no terms corresponding to $n=1$ will appear, since these terms define no distortion.⁸ These series may be used in conjunction with series of solutions of the type given by Eq. (11). Thus the coefficients, and hence the motion, are determined. The motion due to rigid body displacements may, of course, be superimposed on such solutions.

4. Forced vibrations. As will be seen in the following section, the investigation of the possible motions of the constrained ring requires, as a preliminary step, the determination of the behavior of a ring acted upon by a force distribution $N/a = 2A \cos n\theta \cos \lambda s$. The problem involving a driving function P of similar form is obviously covered by this problem. If we split N into two parts,

$$N/a = A [\cos (n\theta - \lambda s) + \cos (n\theta + \lambda s)],$$

a particular solution of the form,

$$v = b_n \cos (n\theta - \lambda s) + c_n \cos (n\theta + \lambda s) \quad (14)$$

is easily shown to exist by substituting this expression into Eq. (9b). The coefficients b_n and c_n are readily found when this is done, and are given by

$$X_n(\lambda) = b_n/A = [n^2(n^2 + 1)(p_n^2 - q_n^2) - (n^2 + 1)\lambda^2 - 4n\mu\lambda]^{-1}, \quad (14a)$$

$$Y_n(\lambda) = c_n/A = [n^2(n^2 + 1)(p_n^2 - q_n^2) - (n^2 + 1)\lambda^2 + 4n\mu\lambda]^{-1}, \quad (14b)$$

unless λ is one of the values given by $\lambda^2 = n^2\beta_n^2$. β_n is either of the values given by Eq. (10a).

When $N/a = 2B \cos n\theta \sin \lambda s$, we have

$$v = d_n \sin (n\theta - \lambda s) - e_n \sin (n\theta + \lambda s),$$

and

$$d_n/B = X_n, \quad e_n/B = Y_n.$$

The quantities X_n and Y_n are useful later in the paper; hence the special notation.

We see now that the motion of the unconstrained ring resulting from the type of loading described is composed of two waves of different amplitudes traveling around the ring with equal but opposite velocities. We note that there is, for each n , one value of λ for which there are fixed nodal points in so far as tangential motion is concerned. This value of λ is defined by $X_n + Y_n = 0$.

It follows from the linearity of our equations that the driving function N of the more general form,

$$N/a = \sum_{m,n} A_{mn} \cos n\theta \sin \lambda_m s \quad (15)$$

will correspond to a solution

$$v = \sum b_{mn} \cos (n\theta - \lambda_m s) + c_{mn} \cos (n\theta + \lambda_m s). \quad (15a)$$

Terms X_{mn} and Y_{mn} are defined as were $X_n(\lambda_m)$ and $Y_n(\lambda_m)$ in Eqs. (14a) and (14b).

The particular problem in which the exciting force is given by

$$N/a = A_0 \cos \lambda s \quad (16)$$

⁸ This is seen in the discussion following the introduction of Eq. (9).

has no inextensional solutions. As was mentioned previously, we must, in this case, use Eq. (9d) for the determination of v . The solution has the form $v = b_0 \cos \lambda s$ where b_0 is given by

$$X_0(\lambda) = \frac{b_0 \lambda^{-2} + \epsilon(1 - \mu^2/\lambda^2)}{A_0 (1 + 3\epsilon\mu^2 - \epsilon\lambda^2)}. \quad (16a)$$

The solution arising from the loading $N/a = A_0 \sin \lambda s$ has the same coefficient. For small λ^2 , $v \doteq A_0 \lambda^{-2} \cos \lambda s$.

The natural frequency for the dilatory type of vibration is found by letting the denominator of Eq. (16a) vanish. Its value is given by

$$\lambda_0' = [(1 + 3\epsilon\mu^2)/\epsilon]^{1/2}.$$

Returning momentarily to the question of accuracy, we note that here as in all subsequent problems the exact values of X_n and Y_n differ from those obtained in this section by terms (in the denominator) of order ϵ . Our work is accurate then when $\lambda \ll \epsilon^{-1/2}$.

5. The supported ring. The first fact to observe in the investigation of the "free vibrations" of the partially constrained ring is that when N and P vanish identically, no solutions to Eq. (9a) which obey the boundary conditions can exist. Specifically, we consider the ring to be supported by a number of evenly spaced, rigid, radial supports (let there be J of them), and suppose the ring to be so fastened to these supports that radial motion is unconstrained at all points, but that $v(2\pi i/J, s)$ must vanish for all values of s and for each integer i . The first part of the appendix is devoted to the outline of a proof that Eq. (9b) has no solution under the foregoing conditions. Since the same proof holds for Eq. (9), we must conclude that the supports exert reactions which are to be accounted for in the differential equation by a function N which does not vanish identically. The problem, physically, becomes that of determining what periodic forces, applied at the supports, are capable of sustaining a motion wherein the supported points of the ring have no displacement (tangentially) at any time. (We assume that the supports must move at precisely the speed ω .) Mathematically, we must determine the eigenvalues λ_i and the corresponding solutions of the differential equation wherein we set $P=0$ and

$$N/a = \left[1 + 2 \sum_{n=J, 2J, \dots}^{\infty} \cos n\theta \right] [A \cos \lambda s + B \sin \lambda s]. \quad (17)$$

This expression defines a loading which must correspond to a motion which has period $2\pi/J$ in θ , since the force is the same at each support. When J is even, there may also be solutions periodic in π/J which don't imply extensional motion. We shall not consider these, however, since both the procedures and results are analogous in the two cases.

We have already shown [Eqs. (15) and (15a)] that for loadings of the type given by Eq. (17), solutions of the form

$$v(\theta, s) = \sum_n [b_n \cos(n\theta - \lambda s) + \dots - e_n \sin(n\theta - \lambda s)] \quad (18)$$

exist for all λ except those for which $\lambda^2 = n^2 \beta_n^2$. Using Eq. (9b) we determined all co-

efficients except those of index zero, and these were found with the aid of Eq. (9d). We may then, in this problem, replace Eq. (9) by the following:

$$L_0(v') = L_0'(N/a), \quad L_1(v - v') = (N - N')/a, \quad (19)$$

where the former equation is merely Eq. (9d), v' and N' are those parts of v and N which are independent of θ , and the latter equation is Eq. (9b). The solutions are now defined by Eq. (18) and the coefficients by Eqs. (14a), (14b), and (16a).

If we now plot for a continuous range of λ ,

$$\sigma(\lambda) = \sum_{n=-\infty}^{\infty} X_n(\lambda) + Y_n(\lambda) = \sum_n (b_n + c_n)/A = \sum_n (d_n + e_n)/B,$$

we find that the resulting graph (Fig. 2) contains two singularities corresponding to each n . Furthermore, there are two values of λ which we may associate with each n for which $\sigma(\lambda)$ vanishes. One of these lies between the two singularities belonging to n ; the other lies to the right of these values. We denote the smaller by λ_n , the larger by λ_n^* .

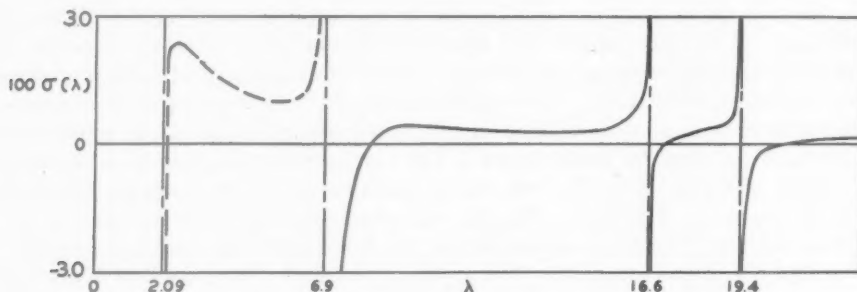


FIG. 2. Response curve for ring driven at two points. $\sigma(\lambda)$ is the amplitude of the motion of the points of application of the force. The broken section of the curve is drawn to the scale 1:5. For this curve, $\mu = 3$.

We observe now that the motion at the points $2\pi i/J$ is given by $v(0, s) = A\sigma(\lambda) \cos \lambda s + B\sigma(\lambda) \sin \lambda s$. The values λ_i and λ_i^* therefore define the frequencies for which the tangential displacements at the points of support are identically zero. They are therefore the desired eigenvalues. The motion of the ring for any λ_i or λ_i^* is given by

$$v_i(\theta, s) = \sum_n A_i X_{in} \cos(n\theta - \lambda_i s) + \cdots + B_i Y_{in} \sin(n\theta - \lambda_i s), \quad (19a)$$

where A_i and B_i are determined by the initial conditions.

The question now arises as to whether linear combinations of the v_i will always describe the motion arising from an initial set of conditions which are arbitrary except for the previously prescribed periodicity in θ . An outline of a proof that the v_i are complete in this sense is included in the appendix. Since the natural modes of the possible vibrations are described accurately by Eq. (19a) only for the smaller values of λ_i , it may seem at first that this question of completeness is superfluous. However, the completeness of such sets of solutions provides an assurance that no other possi-

ble solutions of the equations have been overlooked in the analysis. In view of the fact that the foregoing procedure started with a guess as to the probable form of the support reactions, this indication that certain initial conditions would not lead to different types of motions is helpful. We conclude, then, that all vibrations, arising from initial conditions whose Fourier series expansions are such that the low index terms predominate, can be closely approximated by sums of the form,

$$v(\theta, s) = \sum_i K_i v_i(\theta, s). \quad (20)$$

We also conclude that the supports will continue to exert exactly those reactions required to sustain this motion, or in other words, those forces required to prevent all tangential motion of the supported points of the ring.

The problem of the radially constrained ring may be treated in a manner similar to the foregoing, with analogous results. When the ring is constrained at its supported points against both radial and tangential displacement, Eqs. (9a) and (9b) must both be used. Support reactions of the form

$$N/a = A \left[1 + 2 \sum_n \cos n\theta \right] \cos \lambda s, \quad P/a = B \left[1 + 2 \sum_n \cos n\theta \right] \sin \lambda s,$$

and solutions of the form

$$\begin{aligned} v &= \sum a_n \cos(n\theta - \lambda s) + a'_n \cos(n\theta + \lambda s), \\ u &= \sum b_n \sin(n\theta + \lambda s) - b'_n \sin(n\theta - \lambda s), \end{aligned}$$

are assumed to exist as before. This time we find four functions analogous to $\sigma(\lambda)$ which enter the equations for the motion of the supported points. When this motion vanishes, these equations become $A\sigma_1(\lambda) + B\sigma_2(\lambda) = 0$, $A\sigma_3(\lambda) + B\sigma_4(\lambda) = 0$. The critical frequencies are defined by $\sigma_1(\lambda)\sigma_4(\lambda) - \sigma_3(\lambda)\sigma_2(\lambda) = 0$. Since nothing essentially different from the preceding results would be shown, the formulas for the terms in the σ_i , the explicit expressions for the v_i , etc. are omitted.

The forced vibration problem of the supported ring can now be easily treated. For example, let us consider the ring to be supported as in the first problem of this section of the paper, but to be loaded by a force distribution which may be expanded into the form $N_1/a = \sum_n D_n \cos(n\theta \pm \nu s)$. The particular solution to Eq. (9b) corresponding to the loading N_1 is found as before, and the function

$$v(0, s) = \sum_n g_n \cos \nu s = G \cos \nu s$$

representing the displacement at the supports, has an easily evaluated amplitude G . Remembering that a support reaction,

$$N_2/a = A \left[1 + 2 \sum_n \cos n\theta \right] \cos \nu s$$

produces a motion at $\theta = 2\pi i/J$ which is given by $v_2(0, s) = A\sigma(\nu) \cos \nu s$, we may determine from the response curve (Fig. 2) the value of A such that $A\sigma(\nu) = -G$. The motion is then given by the solution to Eqs. (19) corresponding to the loading $N_1 + N_2$. When more than one value of ν enters the problem, the solution is changed only by the fact that the summation now takes place over two indices.

We note that when ν in the problem just discussed is equal to one of the λ_i , a resonant condition exists, as one would expect from the results of the preceding problem. Also, we note that when the loading consists of a single term, $\cos m\theta \cos \nu s$, and when ν is one of the roots of $X_m + Y_m = 0$, the solution requires no support reactions.

Perhaps the most interesting result of this analysis is the observation that, unlike most problems of this sort, the forces exerted by the rigid supports of the vibrating system must be included in the differential equation before the solution can be obtained.

6. The elastically supported ring. A rather interesting eigenvalue problem arises when we consider the ring with elastic rather than rigid supports. Let us suppose again that the ring is unconstrained radially but that the supports resist the displacement of the points of attachment by a force, $N/a = -Kv(0, s)$.

Using Eqs. (19) as before, we find that the differential equations governing the motion now have the form,

$$L_0(v') = -KL'_0[v(0, s)], \quad (21a)$$

$$L_1(v - v') = -K[v(0, s)][2 \sum \cos n\theta]. \quad (21b)$$

For solutions of the form,

$$v = \sum_n [\alpha_n \cos(n\theta - \lambda s) + \dots - \delta_n \sin(n\theta - \lambda s)],$$

Eqs. (21a) and (21b) become

$$\begin{aligned} \alpha_n &= -KFX_n(\lambda), & \gamma_n &= -KHX_n(\lambda), \\ \beta_n &= -KFY_n(\lambda), & \delta_n &= -KHY_n(\lambda), \end{aligned} \quad (n = 0, J, 2J, \dots). \quad (21c)$$

In these equations, $F = \sum_n (\alpha_n + \beta_n)$, $H = \sum_n (\gamma_n + \delta_n)$, and the X_n and Y_n are again given by Eqs. (14a), (14b) and (16a).

When Eqs. (21c) are added by pairs and then summed over n , the following results are obtained: $F[1 + K\sigma(\lambda)] = 0$, $H[1 + K\sigma(\lambda)] = 0$. But if F and H vanish the solution is the trivial one; hence,

$$\sigma(\lambda) = -1/K. \quad (22)$$

This equation defines the eigenvalues and hence the natural modes at which the system may vibrate. We note that as K tends to infinity λ approaches that value found in the problem of the more strongly constrained ring (as it obviously should). As K tends to zero, the solution approaches that for the unconstrained ring. It is again easily shown that the set of eigen-functions obtained in this problem is complete in the previously used sense.

A final problem in forced vibrations follows easily from the foregoing. Let the ring be supported as above, but with the supports rotating at a speed $\omega + \psi \lambda \sin \lambda s$, where ω is again constant. Briefly, we replace $v(0, s)$ by $v(0, s) + \psi \cos \lambda s$, on the right sides of Eqs. (21a) and (21b). The previously used procedures lead to the familiar set of solutions with the resonant frequencies obviously defined by Eq. (22).

APPENDIX

In this section, we wish to show, first, that Eq. (9b) has no solutions which are periodic in both θ and s and which vanish at $\theta = k\pi/J$, ($k = 0, 1, 2, \dots$, $P = N = 0$).

We assume a solution periodic in θ and write it as the sum of an even and an odd function (in θ);

$$v(\theta, s) = x(\theta, s) + y(\theta, s),$$

where $x(\theta, s) = x(-\theta, s)$ and $y(\theta, s) = -y(-\theta, s)$. In order that v satisfy the specified requirements, both x and y must vanish at the prescribed points.

In an abbreviated form we write Eq. (9b) as $L_2(v) = \partial^2 v / \partial \theta \partial s$, where the operator L_2 transforms even functions into even and odd into odd. This can be seen by inspection of Eq. (9b). The equation is now separable into two parts:

$$L_2(x) = \frac{\partial^2 y}{\partial \theta \partial t}, \quad (a)$$

$$L_2(y) = \frac{\partial^2 x}{\partial \theta \partial t}. \quad (b)$$

We operate on Eq. (b) with the operator $\partial^2(\dots)/\partial \theta \partial t$ and substitute Eq. (a) into the result, obtaining

$$L_2^2(x) = \frac{\partial^4 x}{\partial \theta^2 \partial t^2}, \quad (c)$$

which has solutions of the form, $x = \sum_n a_n \cos n\theta \exp i\gamma_n s$. All even, continuous, periodic, solutions of Eq. (c) may be written in this form but all of such solutions will fail to vanish at the specified points unless the a_n vanish identically, since γ_m/γ_n is irrational and $\cos n\theta$ never vanishes at $\theta=0$. Equation (a) then reduces to $\partial^2 y / \partial \theta \partial t = 0$. It now becomes obvious that the solutions sought do not exist.

Before showing that the functions derived as natural modes of vibration in the section on the supported ring are capable of describing all motions periodic in $2\pi/J$ which arise from arbitrary initial conditions, we introduce the following notation:

$$X_{in} + Y_{in} = \phi_{in}, \quad \lambda_i[-X_{in} + Y_{in}] = \psi_{in}, \quad X_{in}^* + Y_{in}^* = \phi_{in}^*, \dots \quad (d)$$

For any motion described by Eq. (20), the possible initial displacements and velocities may be written

$$v(\theta, 0) = \sum_{i=0}^{\infty} \sum_{n=0, J, \dots}^{\infty} [(A_i \phi_{in} + A_i^* \phi_{in}^*) \cos n\theta + (B_i \phi_{in} + B_i^* \phi_{in}^*) \sin n\theta], \quad (e)$$

$$\frac{\partial v}{\partial s}(\theta, 0) = \sum_i \sum_n [(A_i \psi_{in} + A_i^* \psi_{in}^*) \sin n\theta + (B_i \psi_{in} + B_i^* \psi_{in}^*) \cos n\theta],$$

where the A_i, \dots, B_i^* are to be determined by the initial conditions. However, any initial conditions, periodic in $2\pi/J$ can be written

$$v(\theta, 0) = \sum_n \alpha_n \cos n\theta + \beta_n \sin n\theta, \quad \frac{\partial v}{\partial s}(\theta, 0) = \gamma_n \cos n\theta + \delta_n \sin n\theta. \quad (f)$$

This leads to the relations,

$$\sum_i (A_i \phi_{in} + A_i^* \phi_{in}^*) = \alpha_n, \quad \sum_i (B_i \phi_{in} + B_i^* \phi_{in}^*) = \beta_n, \dots$$

This set of equations may be considered as a group to be solved for the A_i, \dots, B_i^* whenever such solutions exist. Except for special cases, such a set of equations always leads to a unique set of solutions corresponding to each set of $\alpha_n, \dots, \delta_n$. Since Eqs. (f) can express all the specified sets of initial conditions, we see that Eqs. (e) can also accomplish this purpose and hence the v_i are complete in the sense defined above.

The case where i equals zero requires a few additional words. The fact that there is no α_0 or δ_0 seems to imply that we have too few equations for the determination of the A_i, \dots, B_i^* . However, there is only one root of $\sigma(\lambda)$ corresponding to n equal to zero. Hence, the correct correspondence between the $\alpha_i, \dots, \delta_i$ and the A_i, \dots, B_i^* exists.

The proofs outlined in this section are not claimed to be rigorous. They are presented merely to outline the reasoning by which the two hypotheses might be proven if so desired.

NON-LINEAR THEORY OF CURVED ELASTIC SHEETS*

BY

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1. Theories of plane and curved shells which neglect bending.¹ The problem to be treated here is that of determining, under certain conditions to be stated later, the stresses and strains in a thin curved elastic sheet in the form of a surface of revolution held fixed at the edges and subjected to a uniform pressure normal to the sheet. The problem thus falls under the general class of problems treated in the theory of elastic shells.

In order to obtain a theory of elastic shells which is manageable from the mathematical point of view, it is customary to make assumptions² of various kinds, in addition to those of the general theory of elasticity. These additional assumptions are usually based on the hypothesis that the shell is very thin. In what follows we shall be interested in theories which result when the following assumptions are made:

- 1) The strains due to the normal stress on elements parallel to the surface of the shell are small enough to be neglected safely.
- 2) All stresses are constant over the thickness of the shell.

The first assumption is almost always made by writers on the subject of thin shells. The second assumption of course rules out what are usually called bending stresses.

A linear theory of shells, with a considerable number of practical applications, has been worked out on the basis of the above two assumptions. It is usually referred to as the membrane theory of shells.³ The salient feature of the theory is that it is "statically determinate" since the stresses can be obtained from the equilibrium conditions alone without reference to the elastic deformations. This results in a very great simplification, by comparison with theories which do not neglect bending. However, the simplification is coupled with at least one rather serious disadvantage: it turns out that it is not possible to satisfy the kind of boundary conditions which it would be natural to impose in these problems, since the order of the system of differential equations is too low. For example, the condition of a fixed edge (that is, the condition requiring the displacements at the boundary to vanish) cannot be satisfied in general.

Most writers on the membrane theory of shells attribute the difficulty regarding

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¹ The theory developed in this paper is an outgrowth of a research project carried out by the College of Engineering of New York University, under a contract with the War Production Board. The investigation, which was largely experimental in character, was concerned with the feasibility of constructing buildings circular in form with a thin steel roof supported by excess air pressure on the inside of the building. The design problems which arose led to the theory presented here. In this case the sheets considered were so thin that there was no doubt about the validity of neglecting bending stresses.

² For an exhaustive classification of the very numerous possibilities here, see the recent paper of Chien [3]. (Here and in what follows, numbers in square brackets refer to the bibliography at the end of the paper.)

³ For full treatments of this theory and references to the literature, see the books of Flügge [5] and Timoshenko [12].

boundary conditions to the fact that bending is neglected, and it is true that use of the linear bending theory does make it possible to impose physically reasonable boundary conditions. However, there are cases in which the shells are so thin that the bending stresses are small compared with the "membrane" stresses.⁴ It seems not to have been noticed that a theory *which neglects bending stresses, but which nevertheless makes it possible to satisfy physically reasonable boundary conditions, can be obtained by taking account of certain non-linear terms in the relations for the strains as functions of the displacements.* This paper has as its main purpose the development of such a non-linear theory.

Our theory is a generalization of an already existing non-linear theory for the case of a plane sheet⁵ supported in some way at its boundary and subjected to normal pressure p . It is useful for our purposes to discuss the theory of plane sheets from a num-

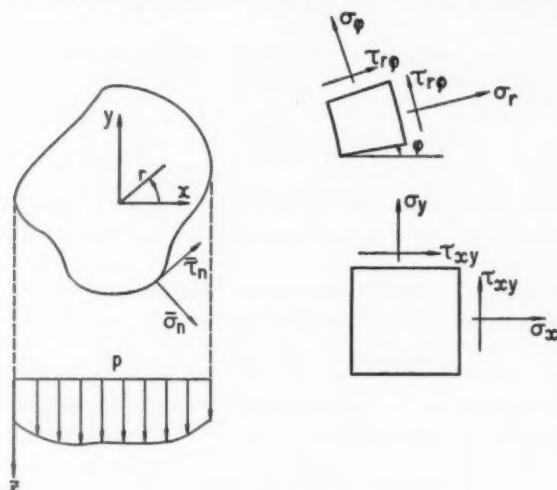


FIG. 1.

ber of different points of view, with the object of comparing and contrasting this theory with the theory of curved sheets to be presented later. The undeformed position of the sheet is taken as the xy -plane, the system of stresses in the sheet is denoted by σ_x , σ_y , and τ_{xy} and the displacement components by u , v , and w . In Fig. 1 the notation for the stresses σ_r and σ_ϕ in polar coordinates (r, ϕ) is also indicated.

The equilibrium conditions for the stresses σ_x , σ_y , and τ_{xy} in the sheet are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \quad (1.1)$$

The equation of equilibrium for the direction normal to the sheet is

$$\sigma_y \frac{\partial^2 w}{\partial x^2} + 2\tau_{xy} \frac{\partial^2 w}{\partial x \partial y} + \sigma_x \frac{\partial^2 w}{\partial y^2} = -p/h, \quad (1.2)$$

⁴ The present investigation was prompted by the necessity of dealing with just such a case.

⁵ The word "sheet" is employed here in a noncommittal way. In the course of our discussion a more precise significance will be given to the phrase "theory of thin sheets."

where h is the thickness of the sheet and p is the pressure. The non-linear character of the theory under discussion stems from the retention of certain quadratic terms in the relations for the strains ϵ_x , ϵ_y , γ_{xy} in terms of the displacements:⁶

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, & \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}.\end{aligned}\quad (1.3)$$

The stresses and strains are assumed, as usual, to obey the stress-strain relations

$$\epsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y), \quad \epsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x), \quad \gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy},$$

where E and ν are the modulus of elasticity and the Poisson ratio, respectively. Finally, we have the "compatibility" equation

$$\nabla^2 (\sigma_x + \sigma_y) = E \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\}, \quad (1.4)$$

in which ∇^2 is the Laplace operator.⁷ This equation is an integrability condition for Eqs. (1.3), expressed in terms of σ_x , σ_y , and w by the use of (1.1) and the stress-strain relations.

Two different kinds of conditions will be considered at the boundary C of the sheet. In one case we prescribe the displacements \bar{u} , \bar{v} , \bar{w} at the boundary;

$$\bar{u} = u(C), \quad \bar{v} = v(C), \quad \bar{w} = 0. \quad (1.5)$$

In the other case, instead of the displacements \bar{u} and \bar{v} we prescribe the normal and shear stresses $\bar{\sigma}_n$ and $\bar{\tau}_n$ at the boundary;

$$\bar{\sigma}_n = \sigma_n(C), \quad \bar{\tau}_n = \tau_n(C), \quad \bar{w} = 0. \quad (1.6)$$

That we may impose the boundary conditions (1.6) is clear; the differential equations (1.1), (1.2), and (1.4) together with the boundary conditions (1.6) constitute the complete formulation of a boundary value problem for the determination of the functions σ_x , σ_y , τ_{xy} and w . That the conditions (1.5) may be imposed could be seen readily by formulating our problem in terms of the displacements u , v , w alone, but we refrain from doing so here. The conditions (1.5) mean that the edge of the sheet is stretched in its plane by a fixed amount, which does not depend upon the applied normal pressure p . The conditions (1.6), on the other hand, mean that the stress at the edge is held fixed while the displacements there will depend upon p .

⁶ Notice that of the three quadratic terms occurring in the usual expression for the strains, only the one involving w is retained in (1.3). The motivation for this is that the order of magnitude of the displacement w normal to the sheet can be expected to differ from that of the displacement parallel to the plane of the sheet. The experimental results (see the paper by Eck [4]) confirm the validity of this assumption from the physical point of view.

⁷ These differential equations were first obtained by Föppl [6] in 1907. They can also be obtained by neglecting the terms referring to bending in the non-linear theory of plates developed by v. Kármán [11]. The equations have been solved by Hencky for the case of a circular sheet [9] and a rectangular sheet [10]. Bourgin [2] has treated the case of the rectangular sheet by methods different from those of Hencky.

For the purpose of comparison with our new theory of curved sheets, which we develop only for the rotationally symmetric case, it is convenient for us to formulate the plane sheet theory in polar coordinates assuming all quantities to depend only on the coordinate r .

The differential equations corresponding to (1.1), (1.2), and (1.4) are for this case,

$$\frac{d}{dr}(r\sigma_r) = \sigma_\phi, \quad (1.1)'$$

$$\frac{1}{r} \frac{d}{dr} \left(r\sigma_r \frac{dw}{dr} \right) = -p/h, \quad (1.2)'$$

$$-\frac{d}{dr}(r\sigma_\phi) + \sigma_r = \frac{E}{2} \left(\frac{dw}{dr} \right)^2. \quad (1.4)'$$

For the purpose we have in view it is not necessary to write down the strain-displacement and stress-strain relations in polar coordinates. The boundary conditions (1.5) at the edge $r=R$ become

$$\bar{u} = u(R), \quad \bar{w} = 0, \quad (1.5)'$$

in which \bar{u} refers to the radial displacement at the edge. The alternate boundary conditions (1.6) become

$$\bar{\sigma}_r = \sigma_r(R), \quad \bar{w} = 0. \quad (1.6)'$$

We consider three different specializations of the non-linear plane sheet theory as a basis for comparison with the theory of curved sheets to be developed later. These are: *Case (a)*, a direct linearization of the differential equations; *Case (b)*, the classical linear membrane theory; *Case (c)*, the problem of Föppl-Hencky. We proceed to discuss these three cases in order.

CASE (a). A direct linearization of the differential equations. If we simply neglect the non-linear terms in (1.2) and (1.3) we obtain the relations $\nabla^2(\sigma_x + \sigma_y) = 0$, $p = 0$. The sheet is therefore not deflected laterally; it is simply in a state of plane stress. From our point of view, such a linearization thus leads to a "trivial" problem.

It is worth while to point out that the solutions for Case (a) are also solutions of the non-linear sheet theory if we impose the condition that the normal pressure p be everywhere zero.

CASE (b). The classical linear membrane theory. The well-known linear theory of tightly stretched plane membranes can be obtained from the non-linear sheet theory as an approximation to the solution of the boundary value problem in a special case. The approximation, as we shall see, results from a development in the neighborhood of Case (a). The special case of the non-linear theory in question arises when the boundary condition is taken in the form (1.6) with $\bar{\sigma}_n$ assumed to be a constant $\bar{\sigma} > 0$, $\bar{\tau}_n$ to be zero:

$$\bar{\sigma}_n = \bar{\sigma} > 0, \quad \bar{\tau}_n = 0, \quad \bar{w} = 0. \quad (1.7)$$

Furthermore, we make the important additional assumption that the applied normal pressure p is small compared with $\bar{\sigma}$. In other words we assume the membrane to be tightly stretched and then deflected by a relatively small normal pressure.

We can solve this boundary value problem by a perturbation method consisting of a development in the neighborhood of the solution for the case in which $w=0$,

$p=0$ (that is, in the neighborhood of the undeflected state of the stretched sheet). The well-known linear membrane theory results as the second step in such a development. We need only develop σ_x , σ_y , τ_{xy} , and w in terms of the parameter ϵ defined by $\epsilon = p/\bar{\sigma}$, as follows:

$$\left. \begin{aligned} \sigma_x &= \sigma_x^{(0)} + \epsilon \sigma_x^{(1)} + \epsilon^2 \sigma_x^{(2)} + \cdots, & \sigma_y &= \sigma_y^{(0)} + \epsilon \sigma_y^{(1)} + \epsilon^2 \sigma_y^{(2)} + \cdots, \\ \tau_{xy} &= \tau_{xy}^{(0)} + \epsilon \tau_{xy}^{(1)} + \epsilon^2 \tau_{xy}^{(2)} + \cdots, & w &= \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \cdots, & p &= \epsilon \bar{\sigma}. \end{aligned} \right\} \quad (1.8)$$

The stresses (including the stress $\bar{\sigma}$ at the boundary) are of lower order in ϵ than the deflection w and the applied pressure p . Insertion of relations (1.8) in the differential equations (1.1), (1.2), (1.4) and the boundary conditions (1.7) leads to a sequence of linear boundary value problems for the determination of the coefficients in the perturbation series. For the terms of zero order in the stresses one finds readily the solution $\sigma_x^{(0)} = \sigma_y^{(0)} = \bar{\sigma}$, $\tau_{xy}^{(0)} = 0$; in other words the zero order terms represent a state of uniform tension throughout the sheet. The zero order terms are also, evidently, the solution for the linearized sheet theory of Case (a). The differential equation for $w^{(1)}$ is then readily found to be

$$\nabla^2 w^{(1)} = -1/h, \quad (1.9)$$

while the boundary condition is, of course,

$$w^{(1)}(C) = 0. \quad (1.10)$$

Equations (1.9) and (1.10) are those of the classical linear membrane theory (for unit normal pressure). For the applicability of this theory the essential condition is that the applied pressure p should be small compared with the initial stress $\bar{\sigma}$ in the sheet. We note also that this theory results when the stress is prescribed at the boundary rather than the displacement in the plane of the sheet; in other words, the linear membrane theory requires that the edge of the sheet be free to move in the xy -plane.

CASE (c). *The problem of Föppl-Hencky.* The boundary value problem which leads to our Case (c) is that resulting from the choice of (1.5) as boundary conditions for the non-linear sheet theory. This theory is sometimes referred to as the large deflection theory of membranes. It is not assumed, as in the above Case (b), that the normal pressure p is small compared with the initial stress in the sheet. In fact, we assume for the Case (c) that the displacements \bar{u} and \bar{v} at the boundary as well as \bar{w} are zero. We shall refer to this problem⁸ on occasion as the problem of Föppl-Hencky. Our boundary conditions of course mean that the sheet was initially unstrained. Thus the stresses in the sheet are built up only as the normal pressure p is applied, and consequently the procedure outlined above for Case (b) is entirely inapplicable.

As already stated, our purpose is to generalize the non-linear sheet theory (c) to the case of curved sheets. The essential step for this purpose consists in developing suitable non-linear strain relations for the curved sheet analogous to those (cf. (1.3)) for the plane sheet. However it is not entirely clear a priori in the case of curved sheets just which of the quadratic terms in the strain equations should be retained and which rejected. Section 2 is devoted to a derivation and discussion of the strain expressions

⁸ The previously cited papers of Hencky [9, 10] and Bourgin [2] are concerned with this problem. The problems for the case in which the boundary displacements \bar{u} and \bar{v} are not zero (i.e. the case of an initially stretched sheet) appear not to have been treated.

used later as the basis for our theory. The discussion is confined to the case of the rotationally symmetric deformation of a surface of revolution. Only two displacements are involved in this case, the displacement u along a meridian and the displacement w along the normal to the sheet.

Once expressions for the strains in terms of the displacements are available, it becomes possible to set up the integral for the potential energy in the sheet (assuming Hooke's law to hold) in terms of the displacements u and w . The equilibrium conditions can then be found as the Euler variational equations minimizing the potential energy. The result is a pair of second order non-linear differential equations for u and w which permit the boundary condition $\bar{u} = \bar{w} = 0$ for a fixed edge to be imposed.

For most purposes it is, however, more convenient to formulate the curved sheet theory in terms of the two components σ_θ and σ_ϕ of the stress in the sheet along and perpendicular to a meridian curve, respectively, and the displacement w normal to the sheet, rather than in terms of the two displacements u and w . This is carried out in Section 3. In Section 4 the general theory is specialized for the case of the spherical sheet. The result is a set of differential equations for the curved sheet analogous to (1.1) and (1.2) for the plane sheet. In Section 5 a simplification in the theory for the spherical sheet is introduced which is valid for a spherical segment of small curvature (and probably also for all cases of spherical sheets). The differential equations of Section 5 are

$$\left. \begin{aligned} \frac{d}{d\theta} (\sigma_\theta \sin \theta) &= \sigma_\phi \cos \theta, \\ \frac{1}{R} \frac{d}{d\theta} \left(\sigma_\theta \sin \theta \frac{dw}{d\theta} \right) &= - \left(\frac{Rp}{h} + \sigma_\phi + \sigma_\theta \right) \sin \theta, \\ - \frac{d}{d\theta} (\sigma_\phi \tan \theta) + (1 + \nu \tan^2 \theta) \sigma_\theta &= \frac{E}{R} \left\{ w \tan^2 \theta + \tan \theta \frac{dw}{d\theta} + \frac{1}{2R} \left(\frac{dw}{d\theta} \right)^2 \right\}. \end{aligned} \right\} \quad (1.11)$$

The independent variable θ is the latitude angle measured from the pole of the sphere. These equations are exactly analogous to Eqs. (1.1)', (1.2)', and (1.4)'.⁹ We refrain from writing the stress-strain and strain-displacement relations which are needed for a complete formulation of the problem.

As boundary conditions at $\theta = \theta_0$ we assume either

$$\bar{u} = u(\theta_0), \quad \bar{w} = w(\theta_0), \quad (1.12)$$

or

$$\bar{\sigma}_\theta = \sigma_\theta(\theta_0), \quad \bar{w} = w(\theta_0). \quad (1.13)$$

At the pole $\theta = 0$ we require all quantities to remain finite.

We wish to consider the three special cases in connection with Eqs. (1.11) which are analogous to the three cases discussed above in connection with the plane sheet. These are: *Case (a)*, a direct linearization; *Case (b)*, the analogue for curved sheets of

⁹ If we were to allow R to tend to infinity while θ tends to zero in (1.11) in such a way that $R\theta \rightarrow r$, the result would be the differential equations (1.1)', (1.2)' and (1.4)'. (The normal pressure p and the displacement w are taken positive in the direction toward the center of the sphere.)

the classical linear membrane theory; *Case (c)*, the analogue of the Föppl-Hencky problem for curved sheets. We consider these cases in order.

CASE (a). *A direct linearization.* In contrast to the corresponding case in the plane sheet theory, we observe that neglect of the non-linear terms in (1.11) does not lead to a trivial problem. We obtain, in fact, the equations

$$\left. \begin{aligned} \frac{d}{d\theta} (\sigma_\theta \sin \theta) &= \sigma_\phi \cos \theta, & \sigma_\theta + \sigma_\phi &= -\frac{Rp}{h}, \\ -\frac{d}{d\theta} (\sigma_\phi \tan \theta) + (1 + \nu \tan^2 \theta) \sigma_\theta &= \frac{E}{R} \left(w \tan^2 \theta + \tan \theta \frac{dw}{d\theta} \right). \end{aligned} \right\} \quad (1.11)'$$

These are the differential equations of what is called the "membrane theory" of thin shells. One observes that the order of the system (1.11)' is two less than that of (1.11). The stresses can be obtained from the first two equations of (1.11)' without reference to the strains and displacements. It is, however, not possible to satisfy in general the kinds of boundary conditions which would be natural in the physical situations encountered in the applications. One such case is that of a fixed edge, which would imply the condition $w=0$ at the boundary. That this condition cannot be satisfied, at least in the case $p=\text{const.}$, is readily seen: the only solution of (1.11)' that is finite at $\theta=0$ is given by

$$\sigma_\theta = \sigma_\phi = -\frac{pR}{2h}, \quad w = pR^2(1-\nu)/2Eh. \quad (1.14)$$

All three quantities are constant throughout the shell.

As in the corresponding Case (a) for the plane sheet, the solution (1.14) of the linear equations (1.11)' is also a solution of the non-linear equations (1.11) if proper restrictions are imposed. Instead of prescribing the boundary values $\bar{\sigma}_\theta$ and \bar{w} in (1.13) arbitrarily, we would require them to have values consistent with (1.14).

CASE (b). *The analogue for curved sheets of the classical linear membrane theory.* The theory of curved sheets analogous to the classical linear membrane theory for plane sheets seems not to have been developed. For the case of a spherical sheet we can obtain such a theory from Eqs. (1.11) with the boundary condition (1.13), in which, however, we assume $\bar{\sigma}_\theta$ and \bar{w} to have values consistent with (1.14) and set $p=p^{(0)}$. However, we assume that the pressure p in (1.11) is given by

$$p = p^{(0)} + \epsilon p^{(1)}, \quad (1.15)$$

in which ϵ is a small (and, of course, dimensionless) parameter. The theory we desire then results from the terms of first order in the development of the solution by perturbations with respect to ϵ . We set

$$\left. \begin{aligned} w &= w^{(0)} + \epsilon w^{(1)} + \epsilon^2 w^{(2)} + \dots, \\ \sigma_\theta &= \sigma_\theta^{(0)} + \epsilon \sigma_\theta^{(1)} + \dots, \\ \sigma_\phi &= \sigma_\phi^{(0)} + \epsilon \sigma_\phi^{(1)} + \dots, \end{aligned} \right\} \quad (1.16)$$

and insert these series together with (1.15) in the differential equations (1.11) and the boundary conditions

$$\bar{\sigma}_\theta = \bar{\sigma}_\phi = -p^{(0)}R/2h, \quad \bar{w} = p^{(0)}R^2(1-\nu)/2Eh. \quad (1.17)$$

The terms of zero order in (1.16) are then readily found to be those which would result from the linear theory—in other words $\sigma_\theta^{(0)}$, $\sigma_\phi^{(0)}$, and $w^{(0)}$ have throughout the sheet the constant values prescribed at the boundary in (1.17). The first order terms are then found to satisfy the linear differential equations

$$\left. \begin{aligned} \frac{d}{d\theta} (\sigma_\theta^{(1)} \sin \theta) &= \sigma_\phi^{(1)} \cos \theta, \\ \frac{1}{R} \frac{d}{d\theta} \left(\sigma_\theta^{(0)} \sin \theta \frac{dw^{(1)}}{d\theta} \right) &= - \left(\frac{Rp^{(0)}}{h} + \sigma_\theta^{(1)} + \sigma_\phi^{(1)} \right) \sin \theta, \\ - \frac{d}{d\theta} (\sigma_\phi^{(1)} \tan \theta) + (1 + \nu \tan^2 \theta) \sigma_\theta^{(1)} &= \frac{E}{R} \left(\tan \theta \frac{dw^{(1)}}{d\theta} + w^{(1)} \tan^2 \theta \right), \end{aligned} \right\} \quad (1.18)$$

and the boundary conditions

$$\bar{\sigma}_\theta^{(1)} = 0, \quad \bar{w}^{(1)} = 0. \quad (1.19)$$

Equations (1.18) and (1.19) are analogous to (1.9) and (1.10) for the corresponding case of the plane sheet. It must, however, be admitted that this "theory of tightly stretched membranes" for the sphere is somewhat artificial because of the fact that the "stretched" state is one for which the initial radial displacement w cannot be held zero at the boundary.

CASE (c). *The analogue of the Föppl-Hencky problem for curved sheets.* The differential equations (1.11) are to be solved for a prescribed pressure p when the edge of the sheet is considered fixed, i.e., under the boundary conditions $\bar{w}=0$ and $\bar{u}=0$. In this particular case the condition $\bar{u}=0$ can be replaced by the condition that the strain in the direction of the boundary curve is zero, which implies the condition $\bar{\sigma}_\phi - \nu \bar{\sigma}_\theta = 0$ on σ_ϕ and σ_θ at the boundary. The analogy with the corresponding case for the plane sheet is, as we see, exact in every respect.

It should now be apparent that some such term as "sheet theory" is needed in addition to the term "membrane theory." This is brought out by Table I which lists the Cases (a), (b), and (c) together with the present terminology. As we note, the phrase "membrane theory" is already applied to cases which have almost nothing in common. Consequently we would recommend (in accordance with a suggestion made by Bourgin [2]) that all of these theories which neglect bending be referred to in

TABLE I.

Cases	Plane	Curved
(a)	Plane stress	Membrane theory of shells
(b)	Membrane with small deflections	—
(c)	Large deflection theory of membranes	—

general as sheet theories. The Cases (a) and (c) could then be referred to as linear and non-linear sheet theories respectively, while the term membrane theory might be reserved for the Cases (b), i.e., for theories of initially stretched sheets which start with

the linear sheet theory as a first approximation and then proceed to a second approximation by a development in the neighborhood of the solution to the linearized problem. This terminology will be used in the remainder of this paper.

In Section 3, the differential equations for the curved sheet theory are obtained, as we have already indicated, by variational methods. In that section also, the stability of the extremal solutions for both the linear and the non-linear sheet theories [Cases (a) and (c)] is considered. At first sight one would be inclined to think that the solutions in the two cases would not differ greatly as far as stability is concerned if the pressure, thickness, etc., are the same in both cases. This is, however, not true. On the basis of the linear curved sheet theory, the solutions would appear to be stable whether the pressure p is inward or outward, that is, whether the sheet is in compression or tension, respectively. On the basis of the non-linear curved sheet theory, however, the solutions are unstable when the pressure is such as to cause the stress σ_θ in the sheet to be a compression.¹⁰ This result follows through consideration of the Legendre condition for our variational problem. In the case of the spherical sheet the stress σ_θ , as given by the linear theory, is a compression when the normal pressure p is positive (i.e., when the pressure is directed toward the center of the sphere). It also seems certain that the non-linear theory will yield the same relation between the signs of σ_θ and p for the case of the spherical sheet, unless the displacements are very large. Consequently, we have assumed in our numerical work that the pressure p is negative, i.e., is directed outward, in order to avoid unstable cases.

In Section 5 the non-linear curved sheet theory [Case (c)] is formulated in detail for the special case of a spherical segment of small curvature. The differential equations for the spherical sheet can be solved by power series in the independent variable. Graphs showing the distribution of the stresses and the normal deflection w along a meridian in a particular numerical case are given in Section 6. Perhaps the most striking feature of these results is that the non-linear sheet theory [Case (c)] yields results which do not differ greatly from those of the linear theory [Case (a)] except near the edge of the sheet. In particular, the stresses and the normal displacement w are nearly constant over most of the interior of the sheet, but change rather rapidly near its edge.

This observation indicates that we have to deal here with a boundary layer effect. In Section (7) the existence of such an effect is deduced and treated explicitly. It turns out upon introduction of proper dimensionless variables in the original differential equations that only one parameter κ remains in the transformed differential equations. The quantity κ is given by

$$\kappa = pR/Eh, \quad (1.20)$$

in which p is the normal pressure on the sheet, R the radius, E the modulus of elasticity, and h the thickness of the sheet. If κ is allowed to tend to zero in the transformed differential equations the result is in the limit the differential equations of the linear sheet theory with a consequent lowering of the order of the system. Hence some boundary condition must be lost at the edge on the transition to the value $\kappa=0$. The solutions of the boundary value problem for $\kappa \neq 0$ can therefore not be expected to converge uniformly at the boundary to the solution of the problem for $\kappa=0$. It is

¹⁰ Comparison with the analogous cases (a) and (c) for the plane sheet theory is illuminating. It is clear that the plane sheet would be stable under edge compression if no lateral deflection were to be permitted, but decidedly unstable under compression if such a lateral constraint were not imposed.

possible to treat the boundary layer phenomenon by introducing a new independent variable which depends upon κ in such a way as to stretch the boundary layer to infinity as $\kappa \rightarrow 0$, with the result that the convergence of the solutions becomes uniform with respect to κ at the edge. One notices that the value $\kappa = 0$ corresponds, according to (1.20), to the value zero for the pressure p .

The boundary layer solution is given in this case very simply by an exponential function. It could be used to estimate the stresses in practice in cases for which κ is small (and, of course, negative). In the usual cases it is not difficult to see that κ will be of the order of -0.0005 in practice, since $pR/2h$ is the stress when p is constant, according to the linear theory, and hence κ is a quantity of the order of the longitudinal strains.¹¹

It is clear that the non-linear sheet theory could be worked out in detail rather readily in other cases such as those of the cylindrical and conical sheets. It would also be of interest to consider the case of the spherical sheet with a hole, so that two distinct boundary curves would exist. Various combinations of boundary conditions at the two edges should be considered; boundary layer effects could then occur at both edges.

From the point of view of the practical applications, another question is of interest. It is clear that bending effects will dominate the "sheet effects" near the edge of the sheet if the sheet is thick enough. This question is under investigation at the present time.

2. Expressions for the longitudinal strains. We assume the curved sheet to be the surface of revolution obtained by rotating about the y -axis the meridian curve C ,

$$x = x(\xi), \quad y = y(\xi). \quad (2.1)$$

The parameter ξ is taken to be the arc length of the curve. We consider only deformations which preserve rotational symmetry, so that the deformation is completely described by the displacement components u and w along the meridian and along the normal to the surface respectively.

It is convenient to introduce the angle θ between the y -axis and the normal to the meridian. These notations are indicated in Fig. 2.

The longitudinal strains in the sheet are defined in the usual way. If ds^1 is the deformed length of the line element originally of length ds , then the strain ϵ in the direction of the element ds is defined by

$$\left(\frac{ds^1}{ds}\right)^2 = 1 + 2\epsilon. \quad (2.2)$$

It is useful to introduce the following relations between the original position (x, y, z) and the deformed position (x^1, y^1, z^1) of any point P on the sheet:

¹¹ The effect of the edge constraint seems to be such as to cause the stresses at the edge to be lower in value than in the interior of the sheet. Thus it seems likely that the usual practice in engineering design of ignoring the edge effect leads to estimates for the stresses which are too high, i.e., are on the side of safety. Of course, we are entitled to draw this conclusion here only in case the sheet is in tension.

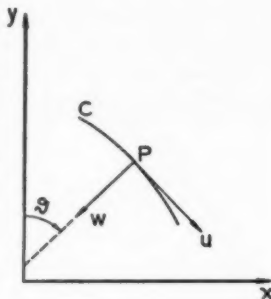


FIG. 2.

$$x^1 = x + u \cos \theta - w \sin \theta, \quad y^1 = y - u \sin \theta + w \cos \theta, \quad z^1 = z.$$

By making use of these relations the strains ϵ_θ and ϵ_ϕ in the direction of a meridian and a parallel (i.e., a curve $\theta = \text{const.}$) are easily computed by using (2.2). We obtain

$$\epsilon_\theta = \frac{1}{\rho} \left(\frac{du}{d\theta} - w \right) + \frac{1}{2\rho^2} \left(u + \frac{dw}{d\theta} \right)^2 + \frac{1}{2} \frac{1}{\rho^2} \left(\frac{du}{d\theta} - w \right)^2, \quad (2.3)$$

$$\epsilon_\phi = \frac{u \cos \theta - w \sin \theta}{x} + \frac{1}{2} \left(\frac{u \cos \theta - w \sin \theta}{x} \right)^2. \quad (2.4)$$

The quantity ρ is the radius of curvature of the meridian curve and x is, of course, the abscissa of the point P .

Just as is done in the analogous case of the plane sheet, we retain only certain of the quadratic terms in the strain expressions, which then amounts to the assumption that these non-linear terms are considered to be of the same order as the linear terms. Thus it would be logical to reject the third term on the right hand side of (2.3) and the second term on the right hand side of (2.4), since they are squares of the linear terms. We shall follow this procedure and thus take for the strains the expressions

$$\epsilon_\theta = \frac{1}{\rho} \left(\frac{du}{d\theta} - w \right) + \frac{1}{2\rho^2} \left(u + \frac{dw}{d\theta} \right)^2, \quad (2.5)$$

$$\epsilon_\phi = \frac{u \cos \theta - w \sin \theta}{x}. \quad (2.6)$$

The following special cases are of interest:

a) *The sphere.*¹² Here $\rho = R$ (the radius of the sphere), and we find from (2.5) and (2.6) that

$$\epsilon_\theta = \frac{1}{R} \left(\frac{du}{d\theta} - w \right) + \frac{1}{2R^2} \left(\frac{dw}{d\theta} + u \right)^2, \quad (2.7)$$

$$\epsilon_\phi = \frac{1}{R} (u \cot \theta - w). \quad (2.8)$$

b) *The circular cylinder.* Here $\rho = \infty$, $\theta = \pi/2$, $x = a$ (the radius of the cylinder), and $\xi = y$. We find that

$$\epsilon_\theta = \frac{du}{dy} + \frac{1}{2} \left(\frac{dw}{dy} \right)^2, \quad (2.9) \quad \epsilon_\phi = -\frac{w}{a}. \quad (2.10)$$

c) *The circular cone.* Here $\rho = \infty$, $\theta = \gamma = \text{const.}$ The strains are given by

$$\epsilon_\theta = \cos \gamma \frac{du}{dx} + \frac{1}{2} \cos^2 \gamma \left(\frac{dw}{dx} \right)^2, \quad (2.11)$$

$$\epsilon_\phi = \frac{u \cos \gamma - w \sin \gamma}{x}. \quad (2.12)$$

¹² These expressions coincide with those used by Friedrichs [7]. Similar expressions were used earlier by Biezeno [1].

3. **Formulation of the boundary value problem for a sheet with fixed edges.** On the assumption that Hooke's law holds for the relation between the strains ϵ_θ and ϵ_ϕ defined above and the corresponding normal stresses σ_θ and σ_ϕ ,¹³ we have for the potential energy V of the sheet the expression

$$V = 2\pi \int_{\xi_1}^{\xi_0} \left(\frac{1}{2} \frac{Eh}{1-\nu^2} [\epsilon_\theta^2 + \epsilon_\phi^2 + 2\nu\epsilon_\theta\epsilon_\phi] - pw \right) x d\xi, \quad (3.1)$$

in which p is the normal pressure on the sheet counted positive in the direction of the inward normal. The quantities ξ and $x(\xi)$ are the arc length and abscissa of the meridian curve, and h is the thickness of the sheet. The quantities ν and E are the elastic constants.

The potential energy could be expressed in terms of the displacements u and w by replacing ϵ_θ and ϵ_ϕ in terms of these quantities through (2.5) and (2.6). The variational equations for the minimum problem would then clearly be a pair of non-linear ordinary differential equations for u and w , each of which would be of the second order. We shall not write these equations down since in the following we wish to work with the stresses σ_θ and σ_ϕ , and the displacement w as dependent variables. However, we do wish to draw one conclusion from the existence of two such equations. The differential equations for u and w are of the proper order to permit imposition of the boundary condition $u=w=0$ appropriate for a fixed edge.

The variational equations resulting from (3.1) are

$$\frac{1}{x} \frac{d}{d\xi} (x\sigma_\theta) - \left[\frac{\sigma_\theta}{\rho} \left(\frac{dw}{d\xi} + \frac{u}{\rho} \right) + \frac{\sigma_\phi \cos \theta}{x} \right] = 0, \quad (3.2)$$

$$\frac{1}{x} \frac{d}{d\xi} \left[x\sigma_\phi \left(\frac{dw}{d\xi} + \frac{u}{\rho} \right) \right] + \left[\frac{\sigma_\theta}{\rho} + \frac{\sigma_\phi \sin \theta}{x} + \frac{p}{h} \right] = 0. \quad (3.3)$$

The quantity ρ in these equations represents the radius of curvature of the meridian curve; the quantities ρ , θ , and x are, of course, given functions of ξ . In deriving (3.2) and (3.3) use was made of the stress-strain relations

$$E\epsilon_\theta = \sigma_\theta - \nu\sigma_\phi, \quad E\epsilon_\phi = \sigma_\phi - \nu\sigma_\theta, \quad (3.4)$$

and of (2.5) and (2.6) in order to introduce σ_θ and σ_ϕ as dependent variables. Equations (3.2), (3.3), (3.4), (2.5), and (2.6), together with appropriate boundary conditions, yield the complete formulation of the boundary value problems we consider here. We note that there are six equations for the six quantities u , w , σ_θ , σ_ϕ , ϵ_θ , ϵ_ϕ .

For the most part, we are concerned with the case of a sheet without a hole at the axis of symmetry, so that the quantity x in (3.2) and (3.3) has the value zero where the meridian curve crosses the axis, which we may assume to occur for $\xi=0$. In this case we would require the solution to be regular at $\xi=0$. At an edge $\xi=\xi_0$ of the sheet we require $u=w=0$, for a fixed edge. In view of (2.6) we see that this implies $\epsilon_\phi=0$; hence we may prescribe the following conditions at a fixed edge:

$$\text{at } \xi = \xi_0 \begin{cases} w = 0, \\ E\epsilon_\phi = \sigma_\phi - \nu\sigma_\theta = 0. \end{cases} \quad (3.5)$$

$$(3.6)$$

In this way we express the boundary condition in terms of w , σ_θ , and σ_ϕ .

¹³ On account of symmetry the shear stresses in the coordinate directions are of course zero.

We remark that the so-called membrane theory of axially symmetric shells results from the above theory when all non-linear terms in σ_θ , σ_ϕ , u , and w are rejected.

We have already stated in the introduction that the solutions of the variational equations (3.2) and (3.3) are unstable when the "radial" stress σ_θ is negative (i.e., when it is a compressive stress). On the other hand, it was stated that the solutions of the linear sheet theory are stable whether σ_θ is positive or negative. The conclusion regarding the instability in the non-linear case results immediately from the fact that the Legendre condition on the second variation of V is not satisfied if σ_θ is negative, which means that the extremals do not render V a minimum in this case. The Legendre condition¹⁴ for a minimum in our case requires that the quantity Δ given by

$$\Delta = F_{u_\xi u_\xi} F_{w_\xi w_\xi} - F_{u_\xi w_\xi}^2 \quad (3.7)$$

should be *positive*. The quantity F is the integrand in (3.1) and subscripts denote differentiations. It turns out that the quantity Δ can be expressed in the form $\Delta = 4\pi^2 h^2 E(1 - \nu^2)^{-1} x^2 \sigma_\theta$. The right hand side has the sign of σ_θ , and consequently the Legendre condition is violated at all points where σ_θ is negative.

In the special case of the spherical sheet, it is possible to put the sign of σ_θ in relation to that of the applied pressure p . If the boundary conditions are specialized in such a way that the solution of the linear sheet theory results, we know [cf. (1.14)] that p and σ_θ are opposite in sign, so that the solutions in this case are unstable when p is positive, i.e., when p is directed toward the center of the sphere. Since it is not possible to give the solutions explicitly in the general non-linear case, we have not been able to prove readily that σ_θ and p are opposite in sign in this case; but if the displacements remain small there can be little doubt that p and σ_θ differ in sign in these cases also. In our further discussion of the spherical sheet we have therefore assumed always that p is negative, i.e., that it is directed outward from the center of the sphere.

The linearized sheet theory results from (3.1) when all terms of degree higher than the second in u and w and their derivatives are neglected at the outset. If this is done, the Legendre condition for the resulting variational problem becomes $\Delta = F_{u_\xi u_\xi} > 0$, with

$$\Delta = \frac{2\pi x E h}{1 - \nu^2}, \quad (3.8)$$

which is always positive, since x (the coordinate measuring the distance from the axis of the sheet) is always positive. Hence the Legendre condition is always satisfied in the case of the linear sheet theory, and we expect all solutions to be stable. The reason for the stable character of all solutions given by the linear theory, as contrasted with the unstable character of some of the solutions given by the non-linear theory, is that the linearization is equivalent to the imposition of a constraint powerful enough to cause stability in all cases.

4. The spherical sheet. In the special case of the sphere we may write $\xi = R\theta$, $p = R$, and $x = R \sin \theta$, R being the radius of the sphere. The differential equations for the sphere are

¹⁴ See, for example, R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. 1, Julius Springer, Berlin, 1931, p. 184.

$$\frac{d}{d\theta}(\sigma_\theta \sin \theta) = \frac{1}{R} \left(\frac{dw}{d\theta} + u \right) \sigma_\theta \sin \theta + \sigma_\phi \cos \theta, \quad (4.1)$$

$$\frac{1}{R} \frac{d}{d\theta} \left[\sigma_\theta \left(\frac{dw}{d\theta} + u \right) \sin \theta \right] = - \left(\frac{Rp}{h} + \sigma_\theta + \sigma_\phi \right) \sin \theta. \quad (4.2)$$

The system of equations is completed by the two strain-displacement relations (2.7) and (2.8) and the stress-strain relations (3.4).

As boundary conditions at a fixed edge $\theta = \theta_0$ we have [cf. the remarks preceding (3.5) and (3.6)]

$$w = 0, \quad (4.3) \quad \sigma_\phi - \nu \sigma_\theta = 0. \quad (4.4)$$

In case the sheet has no hole at the axis, we require the solutions to be regular at $\theta = 0$.

It is of interest to consider the special case of the complete sphere, in which the boundary conditions would become regularity conditions for $\theta = 0$ and $\theta = \pi$. In case we assume that the load p is constant, it is readily verified that a solution of our differential equations which satisfies the regularity conditions is $u = 0$, $w = pR^2(1 - \nu)/2Eh$, $\sigma_\theta = \sigma_\phi = -pR/2h$. It could also be shown that this is the unique solution to this problem. We observe that this solution is identical with that furnished by the linear sheet theory (a) of shells. In other words, the non-linear terms have no influence on the solutions for the full sphere in case the applied pressure p is constant. If p is not constant, however, the non-linear terms will influence the results for the full sphere.

5. The spherical segment with small curvature. The differential equations of the Föppl-Hencky theory for the deflection of a radially symmetric plane sheet are contained in the above equations as a limit case. We need only allow R to tend to infinity while θ approaches zero in such a way that the product $R \sin \theta$ approaches a finite limit r , and r is thus the polar coordinate which locates points in the plane sheet. The resulting equations (1.1)' and (1.2)' have already been given in the introduction. In passing to the limit, one observes particularly that the term u in the second parenthesis on the right hand side of (2.7) drops out, so that the non-linear term reduces to $\frac{1}{2}(dw/dr)^2$. As a consequence of this, the variational equations for the case of the plane sheet are much simpler than (4.1) and (4.2), since the terms corresponding to the first term on the right hand side of (4.1) and the term u in the parenthesis on the left hand side of (4.2) disappear.

It is clear that we could also simplify our equations for the spherical sheet quite considerably by omitting the non-linear terms involving u in the expression (2.7) for ϵ_θ . It would seem fair to expect that such a simplification would be justified for the special case of rather flat spherical sheets. We recall that the choice of the expressions for the strains in terms of the displacements was in any case somewhat arbitrary. At the beginning, we might have considered the displacement u as negligible compared with the quantity $dw/d\theta$, since we expect the order of magnitude of the displacement w in the direction of the applied load to differ from that of the displacement u . In other words, it may well be that this term could be neglected even for sheets of rather large curvature.¹⁵ In what follows we shall neglect this term.

¹⁵ It might be noted that the limit problem which leads to the boundary layer phenomena (to be treated in the next section) is the same whether the terms in the displacement u under discussion here are retained or not. This is another valid reason for considering these terms to be negligible in most cases.

One result of this assumption is that the variational equations no longer contain the function u , but only w , σ_θ , and σ_ϕ . We can obtain a third equation in these same quantities—a “compatibility equation”—by eliminating u from (2.7) and (2.8) and then replacing ϵ_θ and ϵ_ϕ by σ_θ and σ_ϕ through use of the stress-strain relations. The result is the system of equations

$$\frac{d}{d\theta}(\sigma_\theta \sin \theta) = \sigma_\phi \cos \theta, \quad (5.1)$$

$$\frac{1}{R} \frac{d}{d\theta} \left[\sigma_\theta \frac{dw}{d\theta} \sin \theta \right] = - \left[\frac{R\rho}{h} + \sigma_\theta + \sigma_\phi \right] \sin \theta, \quad (5.2)$$

$$-\frac{d}{d\theta}(\sigma_\phi \tan \theta) + (1 + \nu \tan^2 \theta) \sigma_\theta = \frac{E}{R} \left[w \tan^2 \theta + \tan \theta \frac{dw}{d\theta} + \frac{1}{2R} \left(\frac{dw}{d\theta} \right)^2 \right]. \quad (5.3)$$

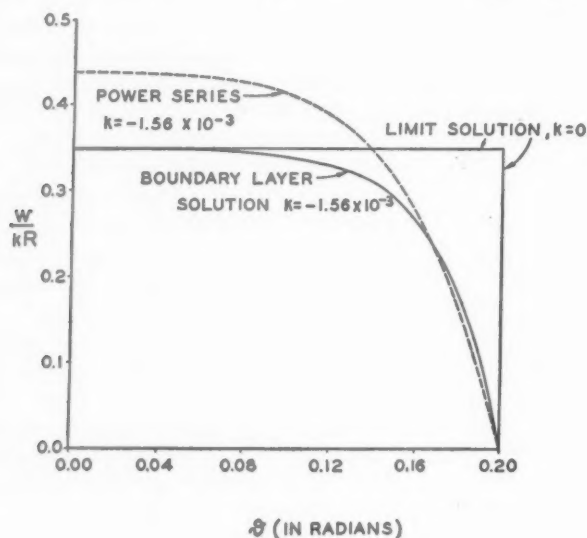


FIG. 3. Normal displacement.

These equations are identical with Eqs. (1.11) which served as the basis for the discussion of the curved sheet theory in the introduction.

We are interested in solving the differential equations (5.1), (5.2), (5.3) for the case of a spherical segment without a hole about the axis $\theta=0$ and with a fixed edge at $\theta=\theta_0$. This means that we require the solution to be regular at $\theta=0$ and to satisfy at the edge $\theta=\theta_0$ the conditions

$$w = 0, \quad (5.4) \quad \sigma_\phi - \nu \sigma_\theta = 0. \quad (5.5)$$

6. Numerical solution of the boundary value problem for the flat spherical segment. Our principal object in this paper is to present a new theory of thin sheets and to compare and contrast it with other theories, rather than to give numerical solutions for the resulting boundary value problems. However, we have obtained approxi-

mate numerical solutions of the boundary value problem formulated in Eqs. (5.1) to (5.5) of the preceding section, and will report the results briefly in this section.

The graphs of Figs. 3, 4, and 5 indicate the results of an approximate solution¹⁸

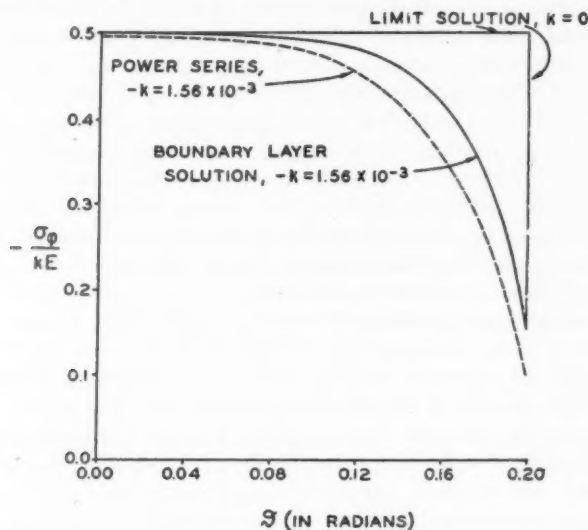


FIG. 4. Circumferential stress.

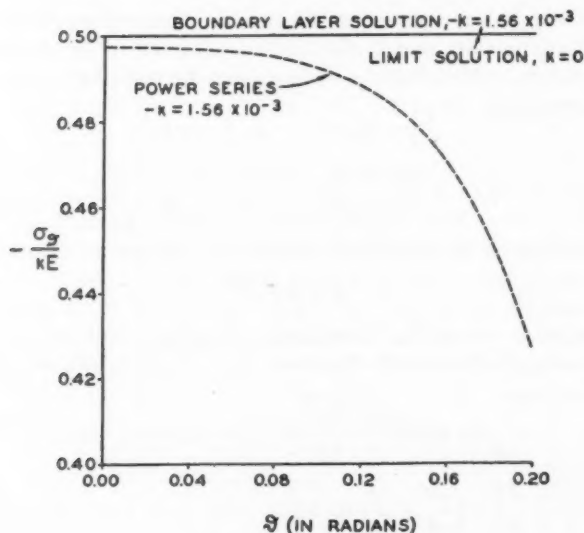


FIG. 5. Radial stress.

¹⁸ The solution was obtained in the form of a development in powers of the independent variable θ . Only four terms in the series (which appear not to converge very rapidly) were retained in calculating coefficients. On the graphs these solutions are marked "power series, $-k=1.56 \times 10^{-3}$."

for the case in which the value θ_0 of θ at the edge of the sheet is 0.2 rad., and the dimensionless parameter¹⁷ $\kappa = pR/Eh$ has the value -1.56×10^{-3} , and $\nu = 0.3$. (This was one of the cases treated as part of the project mentioned in the footnote at the beginning of the paper.) The strain everywhere in the sheet is about $-\frac{1}{2}(1-\nu)\kappa$ in value. The graphs show values of $w/\kappa R$, $-\sigma_\phi/\kappa E$, and $-\sigma_\theta/\kappa E$ as functions of θ . Each graph contains three curves: a dotted curve giving the result of our approximate solution; and two solid curves which refer to the limit solution obtained as $\kappa \rightarrow 0$. The method of obtaining the limit solutions is explained in the next section. One observes that the curves marked "boundary layer solution" approximate those of our numerical solution rather well, at least for the stresses. We have some reason to think that a more accurate solution of the boundary value problem would show the boundary layer solution to be a better approximation to the actual solution than our graphs indicate. We note that the curves marked "limit solution, $\kappa = 0$ " are those which would be obtained from the linear sheet theory.

7. The boundary layer problem. A boundary layer effect has already been mentioned a number of times in connection with our boundary value problem. The graphs of the solutions in the preceding section furnish a hint regarding the character of this phenomenon. The solutions in the interior portion of the sheet appear to be relatively constant, approaching there the values furnished by the linearized theory (i.e., those of the theory usually called the membrane theory of shells). However, toward the edge of the sheet, the solutions appear to change rather abruptly. This is consistent with the repeatedly mentioned fact that the condition for a fixed edge cannot be satisfied in the linearized theory. The purpose of the present section is to treat this boundary layer effect explicitly.

A necessary step in any treatment of boundary layer phenomena¹⁸ consists in the introduction of appropriate new variables and parameters. In the present case it is convenient to introduce new dimensionless dependent variables replacing ϵ_θ , ϵ_ϕ , σ_θ , σ_ϕ , w and u by the relations

$$s_\theta = \sigma_\theta/E\kappa, \quad s_\phi = \sigma_\phi/E\kappa, \quad (7.1)$$

$$e_\theta = \epsilon_\theta/\kappa, \quad e_\phi = \epsilon_\phi/\kappa, \quad (7.2)$$

$$\omega = w/R\kappa, \quad \mu = u/R\kappa, \quad (7.3)$$

in which the important dimensionless parameter κ is defined by the relation

$$\kappa = pR/Eh. \quad (7.4)$$

We assume here that the applied pressure p is constant. In terms of the new quantities, the fundamental differential equations (4.1), (4.2), (2.7), and (2.8) become, in order

$$\frac{d}{d\theta} (s_\theta \sin \theta) = \kappa \left(\frac{d\omega}{d\theta} + \mu \right) s_\theta \sin \theta + s_\phi \cos \theta, \quad (7.5)$$

$$\kappa \frac{d}{d\theta} \left[s_\theta \left(\frac{d\omega}{d\theta} + \mu \right) \sin \theta \right] = - [1 + s_\theta + s_\phi] \sin \theta, \quad (7.6)$$

¹⁷ In the next section it will be seen that θ_0 and κ are the only essential parameters, once the value of the Poisson ratio ν is fixed.

¹⁸ Such boundary layer effects have been well-known for many years in fluid mechanics. They occur also in problems in elasticity other than those considered in this paper. (See, for example [7, 8].)

$$e_\theta = \left(\frac{d\mu}{d\theta} - \omega \right) + \frac{\kappa}{2} \left(\frac{d\omega}{d\theta} + \mu \right)^2, \quad (7.7)$$

$$e_\phi = (\mu \cot \theta - \omega). \quad (7.8)$$

To obtain a complete system of equations we add the stress-strain relations:

$$e_\theta = s_\theta - \nu s_\phi, \quad (7.9) \quad e_\phi = s_\phi - \nu s_\theta. \quad (7.10)$$

As boundary conditions we require all quantities to be regular at $\theta=0$, while at $\theta=\theta_0$ the condition of a fixed edge is prescribed,

$$\omega = 0, \quad (7.11) \quad s_\theta - \nu s_\phi = 0. \quad (7.12)$$

We now observe that if κ is allowed to approach zero in these differential equations, the result is a set of differential equations for the limit quantities which are identically the same as those of the linear sheet theory¹⁹ (when formulated in terms of our dimensionless variables),

$$\frac{d}{d\theta} (s_\theta \sin \theta) = s_\phi \cos \theta, \quad (7.13) \quad s_\theta + s_\phi = -1, \quad (7.14)$$

$$e_\theta = \frac{d\mu}{d\theta} - \omega, \quad (7.15) \quad e_\phi = \mu \cot \theta - \omega. \quad (7.16)$$

Obviously, the boundary conditions (7.11) and (7.12) cannot be imposed in this limit problem. In fact, the solutions of (7.13) and (7.14) are completely determined by the regularity conditions at $\theta=0$ alone. This solution is, as we know, $s_\theta = s_\phi = -\frac{1}{2}$, $\omega = \frac{1}{2}(1-\nu)$, $\mu=0$. In the limit, therefore, the boundary conditions at the edge, in general, will not be satisfied. It follows that the solution of the boundary value problem formulated in (7.5) to (7.12) will not converge uniformly at the boundary to the solution of the limit problem as $\kappa \rightarrow 0$, and this is the essential characteristic of a boundary layer effect.

It is, however, reasonable to expect that the *solutions do converge in the interior* (i.e., for $0 \leq \theta \leq \theta'_0 < \theta_0$, where θ'_0 is a constant) as $\kappa \rightarrow 0$ to the *solutions of the limit problem for $\kappa=0$* . The graphs of the preceding section confirm this to some extent.

It is possible to give an explicit treatment of the boundary layer effect. Such a treatment can be obtained through the introduction of a new independent variable which replaces θ and which depends on κ in such a way that the solutions are made to converge uniformly at the boundary in the limit as $\kappa \rightarrow 0$. What one wants, roughly speaking, is to stretch the boundary layer as $\kappa \rightarrow 0$ in such a way that its width does not shrink to zero. In our case, this can be accomplished by introducing as a new independent variable the quantity η defined by the relation

$$\eta = \frac{1}{\sqrt{-\kappa}} (\theta - \theta_0). \quad (7.17)$$

¹⁹ It is perhaps of interest to observe that the limit situation characterized by $\kappa \rightarrow 0$ can be achieved by allowing the pressure p to approach zero. However, if p is simply set equal to zero in the original differential equations (4.1) and (4.2), the order of the system is not decreased. Thus the introduction of new dependent variables through division of the original ones by κ is an essential step in the treatment of the boundary layer effect.

The reason for the minus sign under the radical is that we wish to consider only cases for which the solutions are stable, which means cases in which κ is negative. (Cf. the remarks at the end of Section 3.)

If we introduce the new independent variable in Eqs. (7.5) to (7.8) and then allow κ to tend to zero, we obtain the set of limit differential equations

$$\frac{ds_\theta}{d\eta} = 0, \quad (7.18) \quad \frac{d}{d\eta} \left(s_\theta \frac{d\omega}{d\eta} \right) = (1 + s_\theta + s_\phi), \quad (7.19)$$

$$0 = \frac{d\mu}{d\eta}, \quad (7.20) \quad e_\phi = \mu \cot \theta_0 - \omega = s_\phi - \nu s_\theta, \quad (7.21)$$

for the range $-\infty < \eta \leq 0$, where $\eta = 0$ corresponds to the edge of the sheet. This system of equations, which has the same order as the original system, yields the boundary layer "resolution" which we seek. The boundary conditions at $\eta = 0$ are given by (7.11) and (7.12). At $\eta = -\infty$ we expect all quantities to tend to the values furnished by the solution of the interior limit problem given above. Thus we expect ω to approach the value $\frac{1}{2}(1 - \nu)$ as $\eta \rightarrow -\infty$.

Since the boundary layer differential equations have constant coefficients, they are readily solved by exponentials. One finds, for example, that ω satisfies the differential equation

$$\frac{d^2\omega}{d\eta^2} - 2\omega = -1 + \nu, \quad (7.22)$$

so that the homogeneous equation is solved by real exponentials.²⁰ The solution of (7.22) which satisfies the conditions at $\eta = 0$ and $\eta = -\infty$ is

$$\omega = \frac{1}{2}(1 - \nu)(e^{\sqrt{2}\eta} - 1). \quad (7.23)$$

The results for the other quantities are easily found to be

$$\mu \equiv 0, \quad (7.24) \quad s_\theta \equiv -\frac{1}{2}, \quad (7.25)$$

$$s_\phi = -\frac{1}{2}\{1 + (1 - \nu)(e^{\sqrt{2}\eta} - 1)\}. \quad (7.26)$$

The graphs of Figs. 3, 4, and 5 contain in each case a curve marked "boundary layer solution, $\kappa = -1.56 \times 10^{-3}$." These curves were obtained from (7.23), (7.25), and (7.26) by reintroduction of θ as a variable through use of (7.17) with $\kappa = -1.56 \times 10^{-3}$. Comparison with the curves for the numerical solution of the original boundary value problem indicates that such a "compressed" boundary layer solution may furnish a fairly good approximation to the values of w and σ_θ near the edge of the sheet if κ is not too large.

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²⁰ If we were to try to resolve the boundary layer in the case when κ is positive (that is, the unstable case) the corresponding equation for ω would possess simple harmonic solutions and no limit would exist as $\eta \rightarrow -\infty$.

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—NOTES—

A FORMULA FOR THE SOLUTION OF AN
ARBITRARY ANALYTIC EQUATION*By D. R. BLASKETT AND H. SCHWERDTFEGER (*University of Adelaide*)

In this note a proof will be given of a formula which has been stated without proof by E. Schröder.¹ The fact that he has expressed some doubts as to its general validity may have caused it to fall into oblivion although it seems to be of some theoretical and practical interest, like several other results of Schröder's to be mentioned below. Since moreover the true nature and simplicity of the formula is rather concealed in Schröder's discussion, it may be worth while to enter into the matter again.

The formula in question is a consequence of the following theorem which at once describes its exact realm of validity:

THEOREM I. *Let $w=f(z)$ be an analytic function, regular in a domain Δ of the complex z -plane. Let α be an interior point of Δ and a simple root of $f(z)$;*

$$f(\alpha) = 0, \quad f'(\alpha) \neq 0.$$

Further let z_0 be a point in Δ "not too far from" α (in practice z_0 is a first rough approximation to the root α). Then, denoting by $z=f^{-1}(w)$ the inverse function of $f(z)$ one has

$$\alpha = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{f(z_0)^{\nu}}{\nu!} \left(\frac{d^{\nu} f^{-1}(w)}{dw^{\nu}} \right)_{w=f(z_0)} = \exp \left(-f(z_0) \frac{df^{-1}(w)}{dw} \right)_{w=f(z_0)} \quad (1)$$

where the exponential function operates symbolically on the differential symbol.

In this form the theorem is a corollary to the main theorem on the analyticity of the inverse of an analytic function. To prove (1) we make use of the fact that because $f'(\alpha) \neq 0$ the function $f(z)$ is simple (schlicht) in a certain neighborhood of the point $z=\alpha$.² Hence the inverse $f^{-1}(w)$ exists in a circle K of radius $\rho > 0$ round the point $w=0$. Moreover it is analytic in K . Thus we may choose z_0 in a neighborhood of α such that a circle k round $w_0=f(z_0)$ (in the w -plane) contains 0 and is wholly contained in K . In k the function $f^{-1}(w)$ is given by its power series

$$f^{-1}(w) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left(\frac{d^{\nu} f^{-1}(w)}{dw^{\nu}} \right)_{w=f(z_0)} \cdot (w - f(z_0))^{\nu}$$

whence for $w=0$ follows the formula (1). Evidently, for z_0 , any point z near α can be chosen for which $w=f(z)$ lies in a circle of radius $\rho/2$ round 0.

To obtain Schröder's formula we introduce the operators δ^{μ} ($\mu=0, 1, 2, \dots$) defined as follows:

* Received Oct. 30, 1944.

¹ E. Schröder, *Über unendlich viele Algorithmen zur Auflösung der Gleichungen*, *Mathematische Annalen* 2, 317-365 (1870); in particular cf. §§4 and 5 of this paper.

² Cf. G. Julia, *Principes géométriques d'analyse*, Première Partie, Paris 1930, p. 16-17.

$$\delta^0 f(z) = \frac{1}{f'(z)}, \quad \delta^1 f(z) = \frac{1}{f'(z)} \frac{d}{dz} \left(\frac{1}{f'(z)} \right), \quad \delta^r f(z) = \frac{1}{f'(z)} \frac{d}{dz} (\delta^{r-1} f(z)).$$

Then

$$\delta^1 f(z) = \left(\frac{d^2 f^{-1}(w)}{dw^2} \right)_{w=f(z)}, \quad \delta^r f(z) = \left(\frac{d^{r+1} f^{-1}(w)}{dw^{r+1}} \right)_{w=f(z)}$$

which is easily shown by induction. Therefore the expansion (1) is identical with

$$\begin{aligned} \alpha &= z_0 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{f(z_0)^{\nu}}{\nu!} (\delta^{\nu-1} f(z))_{z=z_0} \\ &= z_0 - f(z_0) \frac{1}{f'(z_0)} - \frac{f(z_0)^2}{2!} \frac{f''(z_0)}{f'(z_0)^3} + \frac{f(z_0)^3}{3!} \frac{f'(z_0)f'''(z_0) - 3f''(z_0)^2}{f'(z_0)^5} + \dots \end{aligned} \quad (2)$$

This was not the way in which the formula (2) was actually discovered. It appeared rather as a plausible consequence when dealing with another method of approximate solution of an analytic equation. Although no proof could be given on the basis of this method, it turns out to be of some interest here as it shows that the partial sums of the infinite series (1) or (2), i.e., the expressions

$$\Phi_n(z) = \sum_{\nu=0}^n (-1)^{\nu} \frac{f(z)^{\nu}}{\nu!} \left(\frac{d^{\nu} f^{-1}(w)}{dw^{\nu}} \right)_{w=f(z)} = \sum_{\nu=0}^n (-1)^{\nu} \frac{f(z)^{\nu}}{\nu!} \delta^{\nu-1} f(z) \quad (3)$$

can be used for an iterative approximation of the root α ;

THEOREM II. For each $n=1, 2, \dots$ the recurring sequence

$$\alpha_0 = z_0, \quad \alpha_m = \Phi_n(\alpha_{m-1}), \quad (m=1, 2, \dots) \quad (4)$$

has the root α as its limit and then also

$$\Phi_n(\alpha) = \alpha, \quad \Phi'_n(\alpha) = 0, \dots, \Phi_n^{(n)}(\alpha) = 0. \quad (4')$$

This is one of the results of Schröder who has discussed in detail (l.c.) several such iterative algorithms ("Algorithmen erster Art") and has thus obtained formulae of considerable practical interest.³ We propose here another treatment of the problem of iterative approximation which leads immediately to a proof of Theorem II.

We make use of the fact that if α is a simple root of the equation $f(z)=0$, a function $\phi(z)$ can be found for which α is an attractive fixed point, i.e.,⁴

$$\phi(\alpha) = \alpha, \quad |\phi'(\alpha)| < 1. \quad (5)$$

Then the recurring sequence $\alpha_0, \alpha_1 = \phi(\alpha_0), \alpha_2 = \phi(\alpha_1), \dots$, if α_0 is not too far from α , is convergent and has α as its limit. To strengthen the convergence we may replace the inequality in (5) by the condition $\phi'(\alpha)=0$. Such a function $\phi(z)$, involving an arbitrary function $h(z)$, is for instance

³ In particular the reader's attention may be called to the formulae (A_1^h) and (B_2^h) in §12 (p. 352) of Schröder's paper. These formulae are very expedient for the computation of n th roots if a high degree of accuracy (e.g. more than 20 correct figures) is required. A similar algorithm has recently been given by V. A. Bailey in a brief expository article *Prodigious calculation*, Australian Journal of Science, 3, 78-80 (1941), by which our attention has been drawn to the present subject.

⁴ Cf. G. Julia, loc. cit., p. 23.

$$\phi(z) = z + f(z) \frac{f(z)h(z) - 1}{f'(z)}.$$

Thus we may impose more rigid conditions further strengthening the convergence, viz.,

$$\phi'(\alpha) = 0, \phi''(\alpha) = 0, \dots, \phi^{(n)}(\alpha) = 0, \quad (6)$$

where n is any positive integer.⁵

A function $\phi(z)$ satisfying the conditions (5) and (6) can be obtained in the following way. The conditions (6) will be satisfied if the derivative of $\phi(z)$ appears in the form

$$\phi'(z) = (f(z))^n g(z) f'(z),$$

the undetermined function $g(z)$ being regular at $z = \alpha$. It remains to adapt $g(z)$ to the condition $\phi(\alpha) = \alpha$. One has

$$\phi(z) = \int (f(z))^n g(z) f'(z) dz = \int w^n g(f^{-1}(w)) dw$$

whence, by repeated integration by parts, it follows that

$$\phi(z) = n! \sum_{\nu=0}^n \frac{(-1)^\nu}{(n-\nu)!} (f(z))^{n-\nu} g_{\nu+1}(z)$$

where $g_\mu(z)$ is the μ -fold iterated indefinite integral of $g(f^{-1}(w))$ for $w = f(z)$. Thus, for $z = \alpha$ one has

$$\phi(\alpha) = (-1)^n n! g_{n+1}(\alpha).$$

Therefore

$$g_{n+1}(z) = \frac{(-1)^n}{n!} z$$

will give a function $\phi(z)$ which has all the desired properties. In this way one obtains the function $\Phi_n(z)$ of (3), and it is evident that this function has the properties stated in Theorem II.

⁵ From a letter of Professor V. A. Bailey we have learnt (in May 1941) that this problem has been dealt with in some special cases by E. Netto in his *Vorlesungen über Algebra* vol. I, Teubner, Leipzig, 1896, p. 300. In the same letter Bailey has given an elegant solution of the problem which, however, does not suit our present purpose. Further he has drawn our attention to the paper by L. Sancery, *De la méthode des substitutions successives pour le calcul des racines des équations*, Nouvelles Annales d. Math. (2) 1, 305-315 (1862), which, however, was not accessible to us.

THE CAPACITY OF TWIN CABLE*

By J. W. CRAGGS AND C. J. TRANTER (*Military College of Science, Stoke-on-Trent, England*)

1. Introduction. The problem of determining the capacity of two long parallel cylindrical conductors can be easily solved by the use of a conformal transformation.¹ A simple extension of the method gives the result for the case in which each conductor

* Received April 16, 1945.

¹ F. B. Pidduck, *A treatise on electricity*, Cambridge University Press, 1916, p. 77.

is surrounded by a dielectric sheath whose boundary is a member of the coaxial system of circles defined by the boundaries of the conductors. The case in which the sheaths are concentric with the conductors is of much greater practical importance and in many types of cable the sheaths are actually touching. In this paper we give the derivation of the potential distribution for this latter case together with a practical method for the calculation of the capacity.

2. Statement of the problem. We consider the symmetrical problem of two circular wires each of radius R_1 surrounded by concentric touching sheaths of radius R_2 and dielectric constant K_1 , the whole being immersed in an infinite medium of dielectric constant K_2 .

For infinitely long straight wires, the problem reduces to the determination of potentials V_1, V_2 satisfying: (i) the equations

$$\nabla^2 V_1 = 0, \quad (1)$$

for $R_1 \leq r \leq R_2$, and

$$\nabla^2 V_2 = 0, \quad (2)$$

in the region between the circle $r = R_2$ and the line $x = 0$, where ∇^2 is the two dimensional form of Laplace's operator, the polar coordinates r, θ are based on the centre of one of the conductors and the cartesian coordinates x, y have origin at the point

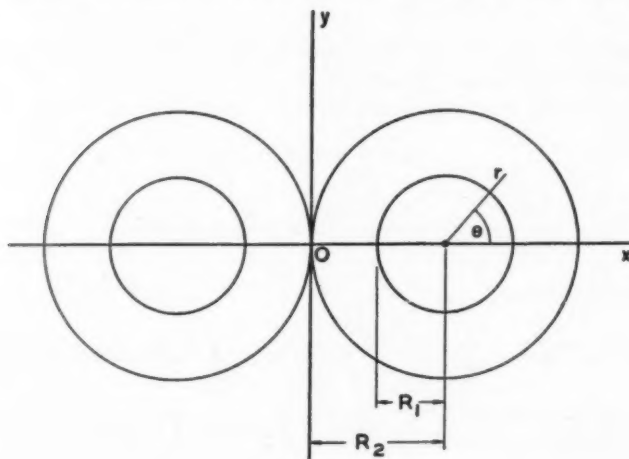


FIG. 1.

of contact of the sheaths and axes as shown in Fig 1; (ii) the boundary conditions

$$V_1 = 1 \quad (3)$$

when $r = R_1$, and

$$V_1 = V_2, \quad (4) \quad K_1 \frac{\partial V_1}{\partial r} = K_2 \frac{\partial V_2}{\partial r}, \quad (5)$$

when $r = R_2$, and

$$V_2 = 0 \quad (6)$$

when $x = 0$. Condition (6) is a result of the symmetry of the problem provided that the potential on the left hand conductor is taken as -1 .

The capacity C per unit length of wire is then given by $C = \frac{1}{2}Q$, where Q is the charge per unit length on either conductor.

The capacity per unit length is unaltered if we replace R_2 by unity and R_1 by $R_1/R_2 (=a)$ and we shall do this in the subsequent work.

3. The analytical solution. It is natural to express the potential V_1 in the polar coordinates defined above. We therefore write

$$V_1 = 1 + B \log \frac{r}{a} + \sum_{n=1}^{\infty} \left\{ \left(\frac{r}{a} \right)^n - \left(\frac{a}{r} \right)^n \right\} b_n \cos n\theta, \quad (7)$$

this being the most general solution of (1), symmetrical in θ and satisfying (3).

Conformal representation by the use of

$$\xi - i\eta = \frac{1}{x + iy} = \frac{1}{1 + re^{i\theta}} \quad (8)$$

transforms the region $r > 1$, $x > 0$ into $0 < \xi < \frac{1}{2}$, the boundaries $x = 0$, $r = 1$ becoming $\xi = 0$, $\xi = \frac{1}{2}$ respectively. The general solution of (2) satisfying (6) and the conditions of symmetry is

$$V_2 = \int_0^{\infty} f(t) \sinh 2\xi t \cos 2\eta t dt. \quad (9)$$

The constants B , b_n of (7) and the function $f(t)$ of (9) are now to be determined from the boundary conditions (4) and (5).

Now on $r = 1$ ($\xi = \frac{1}{2}$) the relation (8) gives

$$\eta = \frac{1}{2} \tan \frac{1}{2}\theta = \frac{1}{2}\beta \quad (10)$$

say, and

$$\frac{\partial V}{\partial r} = -\frac{1}{4} \sec^2 \frac{1}{2}\theta \frac{\partial V}{\partial \xi} = -\frac{1}{4}(1 + \beta^2) \frac{\partial V}{\partial \xi}. \quad (11)$$

Thus (4) and (5) become

$$1 - B \log a + \sum_{n=1}^{\infty} \frac{1 - a^{2n}}{a^n} b_n \cos n\theta = \int_0^{\infty} f(t) \sinh t \cos \beta t dt, \quad (12)$$

$$KB + K \sum_{n=1}^{\infty} \frac{1 + a^{2n}}{a^n} n b_n \cos n\theta = -\frac{1}{2}(1 + \beta^2) \int_0^{\infty} t f(t) \cosh t \cos \beta t dt, \quad (13)$$

where $K = K_1/K_2$.

Multiplying (12) by $\cos n\theta$ ($n = 0, 1, 2, \dots$) and integrating with respect to θ from 0 to π , we have

$$1 - B \log a = \frac{1}{\pi} \int_0^{\pi} \int_0^{\infty} f(t) \sinh t \cos \beta t dt d\theta = \int_0^{\infty} e^{-t} f(t) \sinh t dt \quad (14)$$

since

$$\int_0^{\pi} \cos \beta t d\theta = 2 \int_0^{\infty} \frac{\cos \beta t}{1 + \beta^2} d\beta = \pi e^{-t}$$

and

$$\frac{1 - a^{2n}}{a^n} b_n = \frac{2}{\pi} \int_0^\pi \int_0^\infty f(t) \sinh t \cos \beta t \cos n\theta dt d\theta = \int_0^\infty e^{-t} f(t) \sinh t I_n(t) dt, \quad (15)$$

where

$$\begin{aligned} I_n(t) &= \frac{2e^t}{\pi} \int_0^\pi \cos n\theta \cos \beta t d\theta = \frac{4e^t}{\pi} \int_0^\infty \frac{\cos \beta t}{1 + \beta^2} \cos (2n \tan^{-1} \beta) d\beta \\ &= \sum_{p=0}^{n-1} (-1)^p \frac{n-1 C_p}{(n-p)!} (2t)^{n-p}. \end{aligned} \quad (16)$$

Applying Fourier's integral theorem to (13), we obtain

$$\begin{aligned} -\frac{t}{K} f(t) \cosh t &= \frac{4B}{\pi} \int_0^\infty \frac{\cos \beta t}{1 + \beta^2} d\beta + \frac{4}{\pi} \sum_{n=1}^\infty \frac{1 + a^{2n}}{a^n} n b_n \int_0^\infty \cos n\theta \frac{\cos \beta t}{1 + \beta^2} d\beta \\ &= 2B e^{-t} + e^{-t} \sum_{n=1}^\infty \frac{1 + a^{2n}}{a^n} n b_n I_n(t). \end{aligned} \quad (17)$$

Equations (15) and (17) lead to

$$-\frac{1 - a^{2p}}{K a^p} b_p = B \alpha_p + \sum_{n=1}^\infty \frac{1 + a^{2n}}{a^n} n b_n A_{np}, \quad (18)$$

where

$$\alpha_p = 2 \int_0^\infty e^{-2t} \frac{\tanh t}{t} I_p(t) dt, \quad (19)$$

and

$$A_{np} = \int_0^\infty e^{-2t} \frac{\tanh t}{t} I_p(t) I_n(t) dt. \quad (20)$$

Finally (14) and (17) give

$$\frac{1}{K} (B \log a - 1) = 2B \int_0^\infty e^{-2t} \frac{\tanh t}{t} dt + \frac{1}{2} \sum_{n=1}^\infty \frac{1 + a^{2n}}{a^n} n b_n \alpha_n. \quad (21)$$

The infinite set of equations (18) gives the values of the coefficients b in terms of B . Substitution in (21) yields an equation for B and the capacity per unit length C can then be determined from $C = -\frac{1}{4} K_1 B$, since

$$2C = Q = -\frac{K_1}{4\pi} \int_0^{2\pi} \left(\frac{\partial V_1}{\partial r} \right)_{r=a} a d\theta.$$

This completes the analytical solution.

4. Method of computation. In practice, a good approximation to the capacity may be obtained by retaining only a finite number m of the coefficients b_n . Eliminating the m quantities $[(1 + a^{2n})/a^n] n b_n$ ($n = 1, 2, \dots, m$), from the $(m+1)$ equations (18) and (21), and writing

$$\gamma_n = \frac{1 - a^{2n}}{nK(1 + a^{2n})}, \quad (22)$$

we obtain

$$\begin{vmatrix} A_{11} + \gamma_1 & A_{12} \cdots & A_{1m} & \alpha_1 \\ A_{21} & A_{22} + \gamma_2 \cdots & A_{2m} & \alpha_2 \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} \cdots & A_{mm} + \gamma_m & \alpha_m \\ \alpha_1 & \alpha_2 \cdots & \alpha_m & \frac{2}{K} \left(\frac{1}{B} - \log a \right) + 4 \log \frac{\pi}{2} \end{vmatrix} = 0, \quad (23)$$

since

$$\int_0^{\infty} e^{-2t} \frac{\tanh t}{t} dt = \log \frac{\pi}{2}.$$

The quantities α_n , A_{np} can be easily computed from Eqs. (19), (20) and (16) with the help of the result²

$$\int_0^{\infty} (2t)^n e^{-2t} \tanh t dt = n! \left\{ \left(1 - \frac{1}{2^n} \right) \zeta(n+1) - \frac{1}{2} \right\},$$

where $\zeta(n)$ is the Riemann Zeta-function, tabulated for integral n in J. Edwards, "The integral calculus," vol. 2, Macmillan, London, 1922, p. 144.³

The solution for $K=1$ differs from the well-known exact solution for this case by less than 0.2 per cent, when only the first three of b_n are retained, provided that $a \leq \frac{1}{2}$. For larger values of K and a it may be necessary to retain more terms to achieve the desired accuracy but for practical values the amount of computation required is not excessive.

² When $n=0$, the result reduces to $\int_0^{\infty} e^{-2t} \tanh t dt = \log 2 - \frac{1}{2}$.

³ A four-figure table is given in E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, New York, 1943, p. 273.

LARGE DEFLECTION OF CANTILEVER BEAMS*

By K. E. BISSHOPP AND D. C. DRUCKER (*Armour Research Foundation*)

The solution for large deflection of a cantilever beam¹ cannot be obtained from elementary beam theory since the basic assumptions are no longer valid. Specifically, the elementary theory neglects the square of the first derivative in the curvature formula and provides no correction for the shortening of the moment arm as the loaded end of the beam deflects. For large finite loads, it gives deflections greater than the length of the beam! The square of the first derivative and correction factors for the shortening of the moment arm become the major contribution to the solution of

* Received April 6, 1945.

¹ This problem was considered by H. J. Barten, "On the Deflection of a Cantilever Beam," *Quarterly of Applied Math.*, **2**, 168-171 (1944). Previously an approximate solution had been obtained by Gross und Lehr in *Die Federn*, Berlin VDI Verlag, 1938.

large deflection problems. The following theory which utilizes these corrections is in agreement with experimental observations.

The derivation is based on the fundamental Bernoulli-Euler theorem which states that the curvature is proportional to the bending moment. It is assumed also that bending does not alter the length of the beam.

Considering a long, thin cantilever leaf spring, let L denote the length of beam, Δ the horizontal component of the displacement of the loaded end of the beam, δ the corresponding vertical displacement, P the concentrated vertical load at the free end, B the flexural rigidity, that is $B = EI$, when cross-sectional dimensions are of the

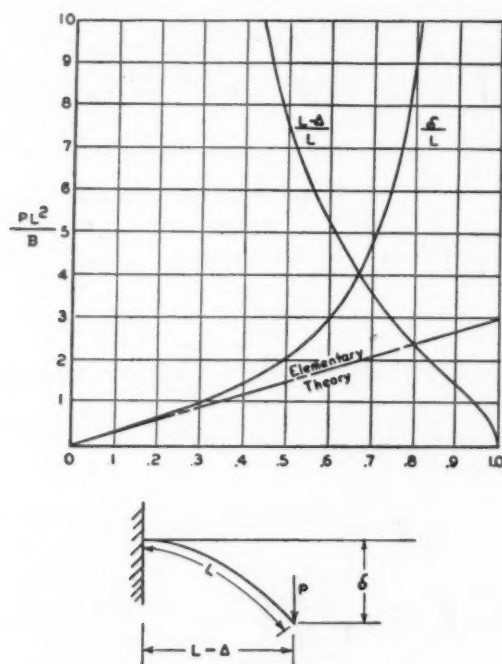


FIG. 1.

same order of magnitude, and $B = EI/(1 - \nu^2)$ for "wide" beams, where ν is the Poisson ratio. The exact expression for the curvature of the elastic line may be stated conveniently in terms of arc length and slope angle denoted by s and ϕ , respectively, so that if x is the horizontal coordinate measured from the fixed end of the beam, the product of B and the curvature of the beam equals the bending moment M :

$$B \frac{d\phi}{ds} = P(L - x - \Delta) = M \quad (1)$$

or

$$\frac{d^2\phi}{ds^2} = -\frac{P}{B} \frac{dx}{ds} = -\frac{P}{B} \cos \phi, \quad (2)$$

whence

$$\frac{1}{2} \left(\frac{d\phi}{ds} \right)^2 = -\frac{P}{B} \sin \phi + C. \quad (3)$$

The constant C can be evaluated directly by observing that the curvature at the loaded end is zero. Then if ϕ_0 is the corresponding angle of slope

$$\frac{d\phi}{ds} = \sqrt{\frac{2P}{B}} (\sin \phi_0 - \sin \phi)^{1/2}. \quad (4)$$

The value of ϕ_0 cannot be found directly from this equation but it is implied by the requirement that the beam be inextensible, so that

$$\sqrt{\frac{2P}{B}} \int_0^L ds = \int_0^{\phi_0} (\sin \phi_0 - \sin \phi)^{-1/2} d\phi = \sqrt{2} \left(\frac{PL^2}{B} \right)^{1/2}. \quad (5)$$

In order to evaluate this elliptic integral, denote PL^2/B by α^2 and let

$$1 + \sin \phi = 2k^2 \sin^2 \theta = (1 + \sin \phi_0) \sin^2 \theta. \quad (6)$$

Then

$$\alpha = \int_{\theta_1}^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta, \quad \sin \theta_1 = \sqrt{2}/2k. \quad (7)$$

The next step is to represent the deflection δ in terms of α and an elliptic integral. Since

$$\frac{dy}{d\phi} \frac{d\phi}{ds} = \frac{dy}{ds} = \sin \phi,$$

and since we have $d\phi/ds$ from Eq. (4),

$$\frac{dy}{d\phi} \sqrt{\frac{2P}{B}} (\sin \phi_0 - \sin \phi)^{1/2} = \sin \phi.$$

Thus

$$\delta = \int_0^y dy = \sqrt{\frac{B}{2P}} \int_0^{\phi} \frac{\sin \phi d\phi}{(\sin \phi_0 - \sin \phi)^{1/2}}.$$

With the aid of Eq. (6) we obtain

$$\frac{\delta}{L} = \frac{\sqrt{2}}{2\alpha} \int_0^{\phi_0} \frac{\sin \phi d\phi}{(\sin \phi_0 - \sin \phi)^{1/2}} = \frac{1}{\alpha} \int_{\theta_1}^{\pi/2} \frac{(2k^2 \sin^2 \theta - 1) d\theta}{(1 - k^2 \sin^2 \theta)^{1/2}}.$$

This equation can be split up into complete and incomplete elliptic integrals of the first and second kinds. In the notation of Jahnke and Emde,

$$\frac{\delta}{L} = \frac{1}{\alpha} [F(k) - F(k, \theta_1) - 2E(k) + 2E(k, \theta_1)], \quad (8)$$

$$\alpha = F(k) - F(k, \theta_1),$$

so that

$$\frac{\delta}{L} = 1 - \frac{2}{\alpha} [E(k) - E(k, \theta_1)]. \quad (9)$$

The horizontal displacement of the loaded end is calculated from Eqs. (1) and (4) with $x=0$ when $\phi=0$. Thus

$$P(L - \Delta) = B \left(\frac{d\phi}{ds} \right)_{\phi=0} = B \sqrt{\frac{2P}{B}} (\sin \phi_0)^{1/2}$$

or

$$\frac{L - \Delta}{L} = \frac{\sqrt{2}}{\alpha} (\sin \phi_0)^{1/2}. \quad (10)$$

From Eq. (6) we have $\sin \phi_0 = 2k^2 - 1$.

Numerical results can be obtained by: (1) selecting values of k corresponding to tabulated values of the modular angle in the elliptic function tables and (2) determining θ_1 and α from Eq. (7). After this has been done, δ/L and $(L - \Delta)/L$ can be calculated from Eqs. (9) and (10) and plotted against $\alpha^2 = PL^2/B$. The results of these calculations are shown in Fig. 1.

CORRECTIONS TO MY PAPER

ON THE DEFLECTION OF A CANTILEVER BEAM*

QUARTERLY OF APPLIED MATHEMATICS, 2, 168-171 (1944)

By H. J. BARTEN

This paper is correct up to the equation

$$\theta_L = \int_0^L a s \cos \theta \, ds.$$

The next step

$$\frac{d\theta_L}{dL} = aL \cos \theta_L$$

is incorrect since θ is not only a function of L , but is also a function of s . This error makes Eqs. (9), (11), and (12) incorrect.

Using the relation

$$\frac{d\theta}{ds} = a(x_L - x)$$

and the various steps used in the original paper, we find that

$$a^{1/2}L = F\left(k, \frac{\pi}{2}\right) - F(k, \delta).$$

By using δ as an independent variable we can calculate corresponding values of k and

* Received June 25, 1945.

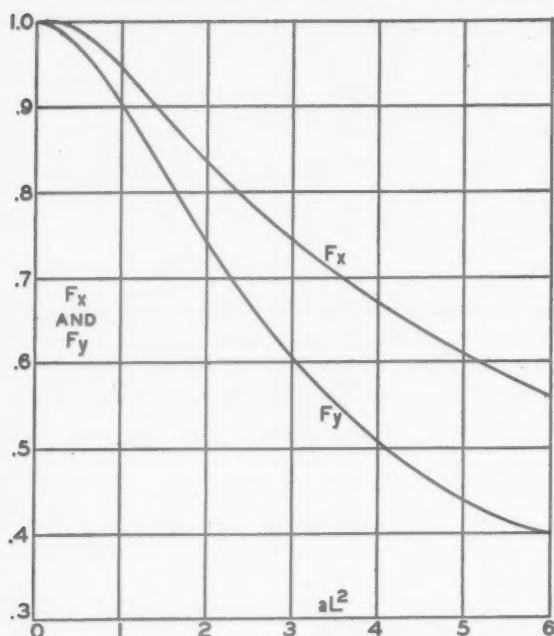


FIG. 1.

aL^2 which in turn are used to find corresponding values of F_x and F_y . The corrected curves thus derived are shown in Fig. 1.

The author wishes to thank M. M. Johnson of Washington, D. C. and D. C. Drucker of the Armour Research Foundation for pointing out these discrepancies.

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